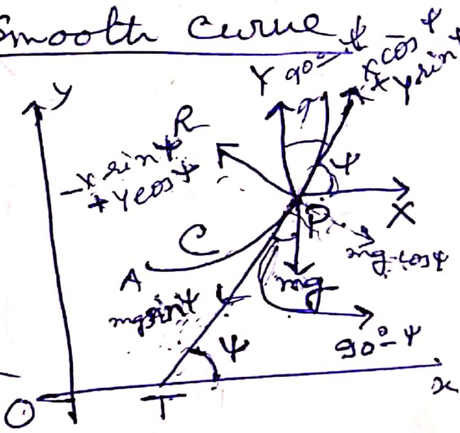
# Constrained Motion



The motion of a particle ~~where~~ is such that the particle is constrained to move in definite curves. ~~These types of motion~~ In this type of motion one considers the accelerations along ~~and~~ the tangent and normal to the given curve.

## Constrained Motion of a Particle on a Smooth Curve




Let a particle of mass $m$ be compelled to move on a given smooth curve C and P be the position of the particle at time $t$.

The dist of P along the curve C from the fixed pt A be $s$, & $v$ be the velocity of the particle at P.

X, Y be the resolved parts of the forces acting at P along the rectangular axes Ox and Oy.

$\because$ The curve is smooth, the reaction ~~of the~~ acting on the particle will be a force R along the normal at P.

Tangent PT ~~ma~~ at P makes angle $\psi$ with the $x$-axis.

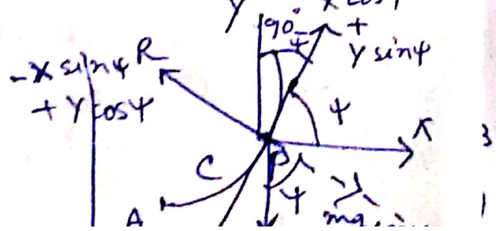
$$\cancel{m\,dv}\quad m v\frac{dv}{ds} = X\cos\psi + Y\sin\psi \quad --(1)$$

$$m\frac{v^2}{\rho} = -X\sin\psi + Y\cos\psi + R \quad -(2)$$

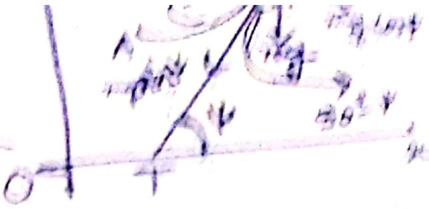
Here only force $mg$ acting vertically downwards

$$\therefore\ m v\frac{dv}{ds} = -mg\sin\psi \quad --(1)$$

$$m\frac{v^2}{\rho} = R - mg\cos\psi \quad --(2)$$



at and P be the position of the particle at time t.



The dist of P along the curve C from the fixed pt A be s, & v be the velocity of the particle at P.

X, Y be the resolved parts of the forces acting at P along the rectangular axes Ox and Oy.

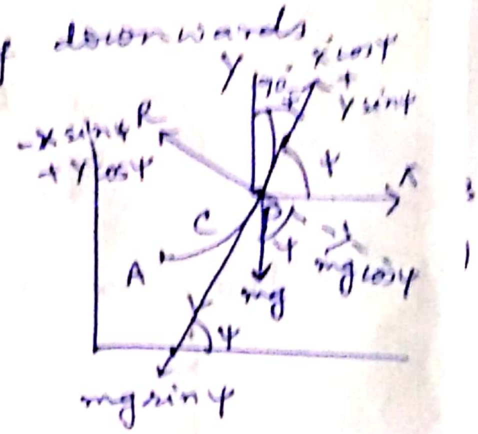
∴ The curve is smooth, the reaction of curve acting on the particle will be a force R along the normal at P.

Tangent PT ~~mo~~ at P makes angle $\psi$ with the x-axis.

$$m v \frac{dv}{ds} = X\cos\psi + Y\sin\psi \quad \text{--- (1)}$$

$$m \frac{v^2}{\rho} = -X\sin\psi + Y\cos\psi + R \quad \text{--- (2)}$$

Here only force mg acting vertically downwards



$$\therefore\ m v \frac{dv}{ds} = -mg\sin\psi \quad \text{--- (1)}$$

$$m \frac{v^2}{\rho} = R - mg\cos\psi \quad \text{--- (2)}$$

$$\sin\psi = \frac{dy}{ds}, \quad \cos\psi = \frac{dx}{ds}.$$

from (1) $\quad m v\, dv = -mg\, dy$

Int. $\quad \frac{mv^2}{2} = C - mgy \quad$ --- (3) $\quad$ C is arb constant.

$v_0$ be the vel. of the particle at $(x_0, y_0)$.

$$\frac{1}{2} m v_0^2 = C - mg y_0 \quad \text{--- (4)}$$

$$\therefore\ \frac{1}{2} m (v^2 - v_0^2) = mg(y_0 - y) \quad \text{--- (5)}$$

from (2) $\quad mv^2 = \rho(R - mg\cos\psi)$

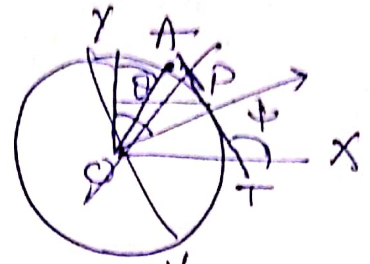
$$\rho R = \rho mg\cos\psi + mv^2$$

$$\& \quad R = mg\cos\psi + \frac{m}{\rho}\left\{v_0^2 + 2g(y_0 - y)\right\} \quad \text{--- (6)}$$

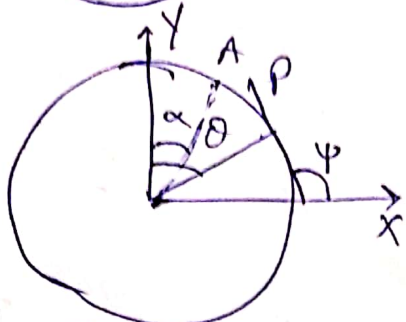
from ~~using~~ the eqn of the curve we can find $\cos\psi$ & $\rho$. R can be determined.

The particle continues to move on the curve as long as R does not vanish and change sign. Equating R = 0 we can find the point on the curve where the particle leaves the curve.

§ Consider the motion of a particle on a smooth circle sliding on the upper side. (2)




P be the position of the particle origin O - centre of the circle.
The particle starts from rest at the pt (at A).




$y = a\cos\theta$, $\qquad v_0 = 0$

$v^2 = \cancel{2ag}\ 2g(y_0 - y)$ from (5)

$= 2ga(\cos\alpha - \cos\theta)$

$R = -\frac{m}{a}\left[2ag(\cos\alpha - \cos\theta)\right] + mg\cos\theta$

$= mg(3\cos\theta - 2\cos\alpha)$

The particle leaves the circle when $R = 0$.

$\therefore \quad 3\cos\theta = 2\cos\alpha$

$\Rightarrow \cos\theta = \frac{2}{3}\cos\alpha$.

& $v^2 = 2ga\left(\frac{3}{3}\cos\alpha - \frac{2}{3}\cos\alpha\right)$

$= 2ga \cdot \frac{\cos\alpha}{3}$

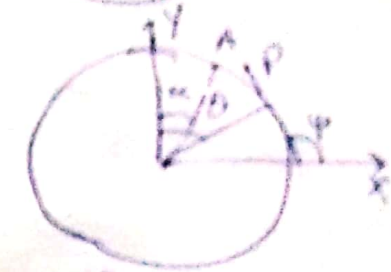
$= \frac{2}{3}ag\cos\alpha$.

Subsequently the particle will move freely under gravity. ~~and describe a parabolic path~~ and describe a parabolic path.

Motion of a particle inside a smooth vertical circle.

smooth ...

P be the position of the particle
origin O - centre of the circle
The particle starts (at A) from rest at the
pt.



$y = a\cos\theta$ . $\quad v_0 = 0$

$v^2 = \cancel{mg}\; 2g(y_0 - y)$ from (5)

$= 2ga[\cos\alpha - \cos\theta)$

$R = -\frac{m}{a}\left[2ag(\cos\alpha - \cos\theta)\right] + mg\cos\theta$ .

$= mg(3\cos\theta - 2\cos\alpha)$

The particle leaves the circle when $R = 0$.

$3\cos\theta = 2\cos\alpha$

$\therefore \theta, \cos\theta = \frac{2}{3}\cos\alpha$ .

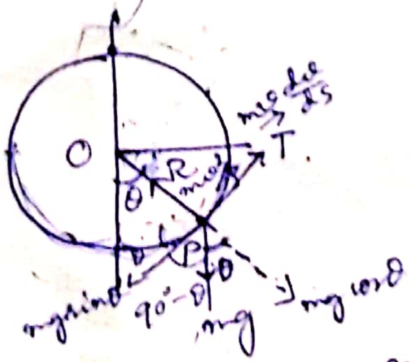
$2 \quad v^2 = 2ga\left(\frac{3}{3}\cos\alpha - \frac{2}{3}\cos\alpha\right)$

$= 2ga \cdot \frac{\cos\alpha}{3}$

$= \frac{2}{3}ag\cos\alpha$ .

Subsequently the particle will move freely under ~~parabolic path~~ .
gravity. ~~and describe a~~
and describe a parabolic path.

Motion of a particle inside a smooth vertical circle.




$mv\frac{dv}{ds} = -mg\sin\theta$ .

$\frac{mv^2}{\rho} = -mg\cos\theta + R$ .

$mv\frac{dv}{d\theta}\cdot\frac{d\theta}{ds} = -mg\sin\theta$

$s = a\theta$ .

$\dot{s} = a\dot{\theta}$

$\ddot{s} = a\ddot{\theta}$

$v\,dv = -ag\sin\theta$ .

$\frac{v^2}{2} = ag\cos\theta + C$

$\frac{ds}{dt} = a\frac{d\theta}{dt}$

Initially $v = u$, $\theta = 0$

$\frac{ds}{d\theta} = a$

$\frac{u^2}{2} = ag + C$

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$$v^2 = u^2 - 2ag(1-\cos\theta).$$

$$R = \frac{m}{a}\{u^2 - 2ag(1-\cos\theta)\} + mg\cos\theta$$

$$= m\left[\frac{u^2}{a} - 2g(1-\cos\theta) + g\cos\theta\right]$$

$$= m\left[\frac{u^2}{a} - 2g + 3g\cos\theta\right].$$

$\theta$ increases from 0 to $\pi$

$\cos\theta$ decreases " 1 to $-1$

At the highest pt $v$ & $R$ are least.

when $\theta = \pi$ $\quad v^2 = u^2 - 4ag$.

$$R = \frac{m}{a}(u^2 - 5ga)$$

If $u^2 > 5ag$, $R$ is +ve. the particle will not leave the ~~plane~~ contact of the circle $v^2 > 0$.

The particle will describe the circle.

$u^2 < 5ag$ then $v$ or $R$ both may be zero before reaching the highest pt.

Let the particle may stop at at pt $\theta = \alpha$ given by putting ~~$\theta$~~. $v = 0$, $\theta = \alpha$

$$\cos\alpha = 1 - \frac{u^2}{2ag}.$$

R vanishes at $\theta = \beta$.

$$\cos\beta = \frac{1}{3}\left(2 - \frac{u^2}{a}\right)$$

$$\frac{u^2}{a} - 2g + 3g\cos\beta = 0$$

$$\cancel{x},\ 2g - \frac{u^2}{a} = 3g\cos\beta$$

$$\cos\beta = \frac{1}{3}\left(2 - \frac{u^2}{a}\right)$$

$\theta$ increases from 0 to $\pi$
$\cos\theta$ decreases " 1 to $-1$
At the highest pt $v$ & $R$ are least.
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$$\cos\alpha = 1 - \frac{u^2}{2ag}$$

$R$ vanishes at $\theta = \beta$.

$$\frac{u^2}{a} - 2g + 3g\cos\beta = 0$$
$$\frac{u^2}{a} \cdot 2g - \frac{u^2}{a} = 3g\cos\beta$$
$$\cos\beta = \frac{1}{3}\left(2 - \frac{u^2}{ag}\right)$$

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If $\alpha < \beta$, then ~~cos~~ $\cos\alpha > \cos\beta$.

$$1 - \frac{u^2}{2ag} > \frac{1}{3}\left(2 - \frac{u^2}{ag}\right)$$

or $\quad 3 - \frac{3u^2}{2ag} > 2 - \frac{u^2}{ag}$

or $\quad 1 > \frac{u^2}{2ag}$ $\qquad$ ie $\cos\alpha > 0$

or $\quad u^2 < 2ag$ $\qquad$ ie in the 1st quadrant

In this case $v$ will be zero at some pt in the 1st quadrant. $R$ will not vanish at any pt during the motion. it will come down, pass through the lowest point and will go up on the left side ie fourth quadrant and rise up to a point having same height and will then come down and so on. It will perform oscillatory motion about the lowest point when $u^2 = 2ag$, $\alpha = \beta = 90°$, the particle will go up to the horizontal level through the centre and will oscillate as before about the lowest pt.

$u^2 = 2ag$.

If $u^2 > 2ag$, ~~$\alpha = \beta = 90°$~~

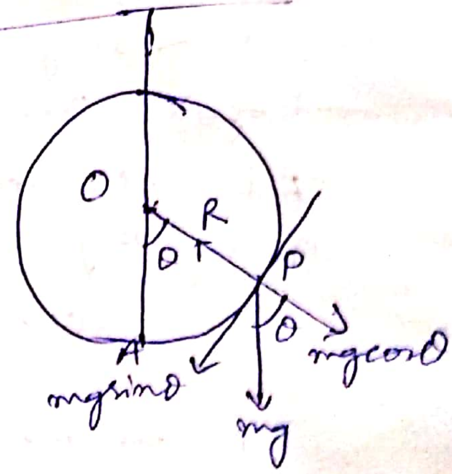
R vanishes before v vanishes since then $\beta > \alpha > 90°$ then the particle will rise above the horizontal diameter. ie $5ag > u^2 > 2ag$., ~~the particle~~ and leave the circle at some pt in the second quadrant and describe a parabolic path.

Ex A particle is free to move on a smooth vertical wire of radius a. It is projected from the lowest point with a velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time $\sqrt{a/g}\log(\sqrt{5}+\sqrt{6})$.

~~Sol~~

Sol O is the centre $\angle AOP = \theta$.

~~Eqn of~~ P ~~be~~ the position of particle at time t.




$AP = s$, $s = a\theta$

Eqn of motions are $\frac{ds}{d\theta} = a$

$$mv\frac{dv}{ds} = -mg\sin\theta \quad \text{--- (1)}$$

$$m\frac{v^2}{a} = R - mg\cos\theta \quad \text{--- (2)}.$$

from (1), $\cancel{m}v\frac{dv}{d\theta}\frac{d\theta}{ds} = -\cancel{m}g\sin\theta$

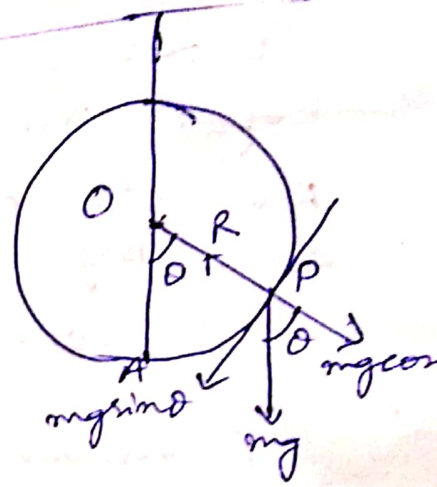
& $v\frac{dv}{a\,d\theta} = -g\sin\theta$

$\ldots\cos\theta + c'$

and ... describe a parabola ...
quadrant and describe a parabola.

Ex A particle is free to move on a smooth vertical wire of radius a. It is projected from the lowest point with a velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time $\sqrt{a/g}\,\log(\sqrt{5}+\sqrt{6})$.

Sol$^n$ O is the centre. $\angle AOP = \theta$.




Eq$^n$ of P be the position of particle at time t.

$AP = s$, $s = a\theta$

Eq$^n$ of motions are $\frac{ds}{d\theta} = a$

$$mv\frac{dv}{ds} = -mg\sin\theta \quad \text{--- (1)}$$

$$m\frac{v^2}{a} = R - mg\cos\theta \quad \text{--- (2)}$$

from (1) $\Rightarrow$ $v\frac{dv}{d\theta}\frac{d\theta}{ds} = -g\sin\theta$

$$v\frac{dv}{a\,d\theta} = -g\sin\theta$$

Int: $\frac{v^2}{2} = +ag\cos\theta + c'$

or $v^2 = 2ag\cos\theta + C$

$v = 0$ at $\theta = \pi$, $A = 2ag$.

$$v^2 = 2ag(1+\cos\theta) \quad \text{--- (3)}$$

from (2) & (3)

$$R = \frac{m}{a}\cdot 2ag(1+\cos\theta) + mg\cos\theta$$

$$= mg(2+3\cos\theta).$$

Let $R = 0$ when $\theta = \alpha$.

$0 = mg\,(2 + 3\cos\alpha)$

or $\cos\alpha = -\frac{2}{3}$

from (8)

$v^2 = 4ag\cos^2\theta/2$

$v = 2\sqrt{ag}\cos\theta/2$

or $a\frac{d\theta}{dt} = 2\sqrt{ag}\cos\theta/2$ $\qquad \because s = a\theta$. $\frac{ds}{dt} = a\frac{d\theta}{dt}$

$\therefore \frac{d\theta}{dt} = 2\sqrt{\frac{g}{a}}\cos\theta/2$

or $2\sqrt{\frac{g}{a}}\,dt = \sec\theta/2\,d\theta$

$\theta = 0$ when $t = 0$, $\theta = \alpha$, $t = t_1$

$2\sqrt{g/a}\int_0^{t_1} dt = \int_0^{\alpha} \sec\theta/2\,dt$

or $2\sqrt{g/a}\,t_1 = 2\left[\log(\sec\theta/2 + \tan\theta/2)\right]_0^{\alpha}$

or $t_1 = \sqrt{\frac{a}{g}}\log(\sec\alpha/2 + \tan\alpha/2)$

$\cos\alpha = -\frac{2}{3}$, $\quad 2\cos^2\alpha/2 - 1 = -2/3$

$2\cos^2\alpha/2 = \frac{1}{3}$, $\quad \cancel{\sin^2\alpha/2 = 5/6}$

$\sec^2\alpha/2 = 6$, $\quad \tan^2\alpha/2 = 6 - 1 = 5$

$\sec\alpha/2 = \sqrt{6}$ $\qquad \tan\alpha/2 = \sqrt{5}$

$t_1 = \sqrt{\frac{a}{g}}\log(\sqrt{5} + \sqrt{6})$.

H.W. Submit the solution on 24.3.2020 at 8-30 a.m.