Vector space

Let V be a nonempty set and $\oplus$ be a binary composition on V. Let $(F, +, \cdot)$ be a field and let $\odot$ be an external composition of F with V.

[$\oplus$ is a mapping from $V \times V$ to V, $\odot$ is a mapping from $F \times V$ to V and it maps the ordered pair (c, α) of $F \times V$ to a definite element $c \odot \alpha$ in V.]

V is said to be a vector space (or a linear space) over the field F if the following conditions are satisfied.

V_1. $\alpha \oplus \beta \in V$ for all $\alpha, \beta \in V$

V_2. $\alpha \oplus \beta = \beta \oplus \alpha$ for all $\alpha, \beta \in V$

V_3. $\alpha \oplus (\beta \oplus \nu) = (\alpha \oplus \beta) \oplus \nu$ for all $\alpha, \beta, \nu \in V$

V_4. there exists an element θ in V such that $\alpha \oplus \theta = \alpha$ for all $\alpha \in V$;

V_5. for each α in V there exists an element $-\alpha$ in V such that $\alpha \oplus (-\alpha) = \theta$

V_6. $c \odot \alpha \in V$ for all $c \in F$, all $\alpha \in V$;

V_7. $c \cdot (d \odot \alpha) = (c \cdot d) \odot \alpha$ for all $c, d \in F$ for all $\alpha \in V$.

V_8. $c \cdot (\alpha \oplus \beta) = (c \odot \alpha) \oplus (c \odot \beta)$ for all $c \in F$, all $\alpha, \beta \in V$

V_9. $(c+d) \odot \alpha = (c \odot \alpha) \oplus (d \odot \alpha)$ for all $c, d \in F$ all $\alpha \in V$.

V_{10}. $1 \odot \alpha = \alpha$ 1 is the identity element in F.

V_1. $\alpha\ (\alpha_1, \alpha_2, \alpha_3 - - - \alpha_n)$

$\beta\ (\beta_1, \beta_2, \beta_3 - - - \beta_n)$

$\alpha \oplus \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3 - - - , \alpha_n + \beta_n) \in V$.

V_2. $= (\beta_1 + \alpha_1, \beta_2 + \alpha_2, \beta_3 + \alpha_3 - - - , \beta_n + \alpha_n) \in V$

$= \beta \oplus \alpha$.

V_3. $\rightarrow$ associative prop. is satisfied - [similarly]

V_4 $\theta = (0, 0, 0, \ldots, 0)$

$\alpha \oplus \theta = (\alpha_1, \alpha_2, \ldots, \alpha_n) + (0, 0, \ldots, 0)$

$= (\alpha_1 + 0, \alpha_2 + 0, \ldots, \alpha_n + 0)$

$= (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$= \alpha.$

V_5. $\alpha \oplus (-\alpha) = (\alpha_1 - \alpha_1, \alpha_2 - \alpha_2, \ldots, \alpha_n - \alpha_n)$

$= (0, 0, \ldots, 0).$

$= \theta.$

$c \in F$, $\alpha \in V$.

V_6. $c \odot \alpha = c \odot (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$= (c\alpha_1, c\alpha_2, \ldots, c\alpha_n) \in V.$

V_7. $c \odot (d \odot \alpha) = c \odot (d\alpha_1, d\alpha_2, \ldots, d\alpha_n)$

$= (cd\alpha_1, cd\alpha_2, \ldots, cd\alpha_n)$

$= \cancel{cd(\alpha_1} \; cd \odot (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$= (c \cdot d) \odot \alpha.$

V_8. $c \odot (\alpha \oplus \beta) = (c \odot (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)$

$= (c\alpha_1 + c\beta_1, c\alpha_2 + c\beta_2, \ldots, c\alpha_n + c\beta_n)$

$= c\alpha \oplus c\beta.$

$= (c \odot \alpha) \oplus (c \odot \beta).$

V_9 & V_{10} H.W.

Th. In a vector space V over a field F,

i) $0\alpha = \theta$ for all $\alpha \in V$;

ii) $c\theta = \theta$ " " $c \in F$;

iii) $-1\alpha = -\alpha$ " " $\alpha \in V$, 1 being the identity element in F.

iv) $c\alpha = \theta$ implies either $c = 0$ or $\alpha = \theta$

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Proof i) 0 is the zero element in F.

$0 + 0 = 0$ in F

$\Rightarrow (0+0) \odot \alpha = 0 \odot \alpha$ in V

$\Rightarrow 0 \odot \alpha + 0 \odot \alpha = 0 \odot \alpha$ by (ii) V_7

$\Rightarrow -0 \odot \alpha \in V$ by (iv) V_5 $\quad$ $\because 0 \odot \alpha \in V$, $-0 \odot \alpha \in V$.

$\therefore -0 \odot \alpha + (0 \odot \alpha + 0 \odot \alpha) = -0 \odot \alpha + 0 \odot \alpha$

or $(-0 \odot \alpha + 0 \odot \alpha) + (0 \odot \alpha) = \theta$

or $0 \odot \alpha = \theta$ $\quad$ $[0 \odot \alpha = 0\alpha]$.

ie $0\alpha = \theta$.

ii) θ is the zero element in V.

$\theta + \theta = \theta \in V$. $\quad$ $c \odot \theta = c\theta$.

$\Rightarrow c(\theta + \theta) = c\theta$.

$\Rightarrow c\theta + c\theta = c\theta$. $\quad$ $c\theta \in V$, $-c\theta \in V$.

$-c\theta + (c\theta + c\theta) = -c\theta + c\theta$

or $(-c\theta + c\theta) + c\theta = \theta$

or $\theta + c\theta = \theta$

or $c\theta = \theta$.

iii) $\theta = 0\alpha$ $\quad$ $1\alpha = 1 \odot \alpha$

$= [1 + (-1)]\alpha$

$= 1\alpha + (-1)\alpha$

$= \alpha + (-1)\alpha$

$-\alpha + \theta = -\alpha + \{\alpha + (-1)\alpha\}$

$= (-\alpha + \alpha) + (-1)\alpha$

$= \theta + (-1)\alpha$

$\therefore -\alpha = (-1)\alpha$.

iv) $c\alpha = \theta$ and let $c \neq 0$. Then c^{-1} exists in F. [$\because$ F is a field].

$c\alpha = \theta \Rightarrow$

or, $c^{-1}(c\alpha) = c^{-1}\theta$

or $(c^{-1}c)\alpha = c^{-1}\theta$

or $1\alpha = \theta$

$\therefore$ $c\alpha = \theta$ and $c \neq 0 \Rightarrow \alpha = \theta$.

contrapositively, $c\alpha = \theta$ and $\alpha \neq \theta \Rightarrow c = 0$.

Hence $c\alpha = \theta \Rightarrow$ either $c = 0$ or $\alpha = \theta$.

Vector Subspace :- Let V be a vector space over a field F with respect to $\oplus$ & multiplication $\odot$ by elements of F. Let W be a nonempty subset of V. If W forms a vector space over F with respect to $\oplus$ and $\odot$, then W is said to be a sub-vector space or a linear subspace or a subspace of V.

Theorem. A non-empty subset W of a vector space V over a field F is a subspace of V if and only if

i) $\alpha \in W, \beta \in W \Rightarrow \alpha + \beta \in W$; and ii) $\alpha \in W$, $c \in F \Rightarrow c\alpha \in W$.

Let the conditions hold in W.

Proof :- Let the conditions hold in W. Let $\alpha, \beta \in W$, since F is a field, $-1 \in F$ where 1 is the identity element in F.

by ii) $-1\beta \in W$ ie $-\beta \in W$.

Then by i) $\alpha + (-\beta) \in W$ ie $\alpha - \beta \in W$.

Thus $\alpha, \beta \in W$, $\alpha - \beta \in W$.

This proves that W is a subgroup of the additive group V.

V is a commutative group, W is also a commutative subgroup of V.

$\therefore$ ~~$V_1 - V_5$~~ The conditions $V_1 - V_5$ for a ~~subspace~~ vector space are satisfied in W.

V_6 is satisfied in W by (ii)

$V_7 - V_{10}$ are satisfied in W, they are hereditary properties.

$\therefore$ W is by itself a vector space over F and so W is a subspace of V.

Ex 1 Let R^3 be a the vector space of all 3-tuples of real numbers. Then $W = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$ is a subspace of R^3.

Solution :- i) If $(a_1, a_2, 0) \in W$ and $(b_1, b_2, 0) \in W$.

Then $(a_1, a_2, 0) + (b_1, b_2, 0) = (a_1 + a_2, b_1 + b_2, 0) \in W$.

ii) If $c \in R$ and $(x, y, 0) \in W$ then
$c(x, y, 0) = (cx, cy, 0) \in W$.

$\therefore$ W is closed for addition and scalar multiplication.

$\therefore$ W is a subspace of R^3.

Ex 2 which of the following sets of vectors $\alpha = (a_1, a_2, \cdots a_n)$ in R^n are subspaces of R^n $(n \geq 3)$, for all α such that

i) $a_1 \geq 0$, ii) $a_1 + 3a_2 = a_3$

iii) $a_2 = a_1^2$ iv) $a_1 a_2 = 0$ v) a_2 is rational,

i) Let $W = \{\alpha : \alpha \in R^n$ and $a_1 \geq 0\}$

Let $\alpha = (a_1, a_2, \cdots, a_n)$ $\alpha, \beta \in W$, $a_1, b_1 \geq 0$

$\beta = (b_1, b_2, \cdots, b_n)$ ~~$a_i, b_i \geq 0$~~.

~~$i = 1, 2, \cdots, n$~~.

Th The necessary and sufficient condition for a non-empty sub-set W of a vector space V over F to be a subspace of V is $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

Proof The condition is necessary

Let W be a subspace of a vector space V. Then W is ~~itsof~~ itself a vector space over the field F. So, by defn of vector space

$a \in F, \ \alpha \in W \Rightarrow a\alpha \in W$

$b \in F \quad \beta \in W \Rightarrow b\beta \in W$

$\Rightarrow a\alpha + b\beta \in W$.

The condition is sufficient :

Suppose W is a non empty subset of V satisfying the given condition ie $a, b \in F$ ~~Taking $a = b = 1$~~

~~and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$~~ (∵ $\alpha \in W \Rightarrow \alpha \in V$ and ~~$1\alpha \in \alpha \in V$~~).

~~$1.\alpha + 1.\beta \in W \Rightarrow \alpha + \beta \in W$~~

~~W is closed under vector addition~~.

Take $a = b = 1$.

$a\alpha + b\beta = 1.\alpha + 1.\beta \in W$.

$\Rightarrow \alpha + \beta \in W$

∴ W is closed under vector addition.

Taking $a = -1, \ b = 0, \quad (-1)\alpha + b\alpha \in W$

$\Rightarrow (-1)\alpha + 0 \in W$

$\Rightarrow -\alpha \in W$. [additive inverse of each element of W is in W]

Take $a = 0, b = 0, \quad 0.\alpha + 0.\beta \in W$

$\Rightarrow$ ~~$0 \in W$~~. $0 \in W$.

∴ zero vector ~~tod~~ of V belongs to W.

$\Rightarrow$ it ~~is~~ also zero vector in W.

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∵ Elements of W are also elements of V so vector addition is commutative as well as associative in W.
∴ W is an abelian group w.r.t vector addition.
"Furthermore, W is closed under scalar multiplication. The remaining postulates of a vector space will hold in W since they hold in V, of which W is a subset.
∴ W itself is a vector sub-space of V.

<u>Solution of Ex 2</u>

i) Let $W = \{\alpha : \alpha \in R^n,\ a_1 \geq 0\}$

$\alpha = (a_1, a_2, \dots a_n)$ $\quad \alpha, \beta \in W$

$\beta = (b_1, b_2, \dots, b_n)$ ~~ew~~ with $a_1, b_1 \cancel{\geq} 0$

For, $a, b \in R$, we have

$a\alpha + b\beta = (aa_1 + bb_1, aa_2 + bb_2, \dots, aa_n + bb_n) \notin W$

because if $a = -2$, $b = -1$, $a_1 = 3$, $b_1 = 2$

Then $aa_1 + bb_1 = -6 - 2 = -8 < 0$

ie First component of the vector $a\alpha + b\beta$ is less than zero

∴ W is not a subspace of R^n.

ii) Let $W = \{\alpha : \alpha \in R^n$ and $a_1 + 3a_2 = a_3\}$

$\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n) \in W$.

$a_1 + 3a_2 = a_3$.

$b_1 + 3b_2 = b_3$.

Now for $a, b \in R$ we have

$(aa_1 + bb_1) + 3(aa_2 + bb_2)$

$= a(a_1 + 3a_2) + b(b_1 + 3b_2)$

$= aa_3 + bb_3$

∴ $a\alpha + b\beta \in W$. W is a subspace of R^3.

iii), iv), v) — H.W. Submit in WhatsApp group on 23.3.2020 at 8-30 a.m.