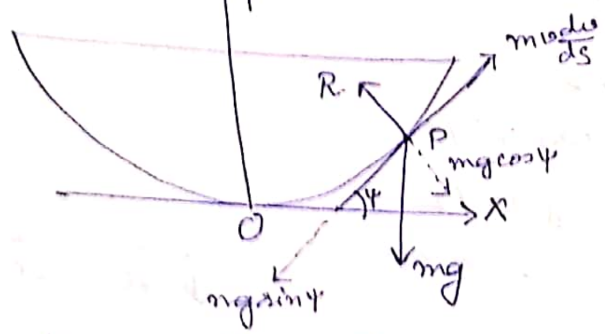
23/3/20

Motion in a ~~S~~mooth cycloid under Gravity:



eq^n of cycloid,

$s = 4a\sin\psi$ - - ①

a = radius of generating circle.

$\rho = \frac{ds}{d\psi} = 4a\cos\psi$

At any time P be the position of the particle.

eq^n of motion

$$\cancel{m} v\frac{dv}{ds} = -\cancel{m}g\sin\psi \quad \cdots ②$$

$$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \cdots ③$$

from ②
$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s \quad \cdots ④$$

from ④ → the motion of the cycloid is simple harmonic having a period $\frac{2\pi}{\sqrt{g/4a}} = 2\pi\sqrt{\frac{4a}{g}} = 4\pi\sqrt{\frac{a}{g}}$ which is independent of arc length.

from ②
$$v\frac{dv}{ds} = -\frac{g}{4a}s$$

Integrating
$$\frac{v^2}{2} = -\frac{g}{4a}\frac{s^2}{2} + \frac{c}{2}$$

or
$$v^2 = -\frac{g}{4a}s^2 + C$$
$$= -\frac{g}{4a}(4a\sin\psi)^2 + C$$

The particle starts from rest at $\psi = \psi_0$

$\cancel{0} = +\frac{g}{4a}$ ∴ $C = 4ag\sin^2\psi_0$

$$v^2 = 4ag(\sin^2\psi_0 - \sin^2\psi)$$

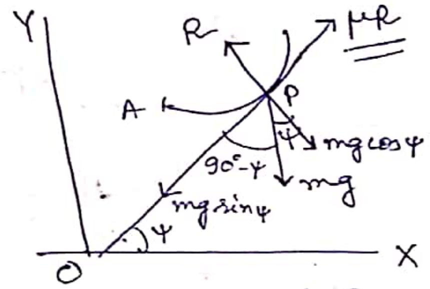
$$R = mg\cos\psi + \frac{m}{4a\cos\psi}\cdot 4ag(\sin^2\psi_0 - \sin^2\psi)$$

$$= mg\cos\psi + mg\sec\psi(\sin^2\psi_0 - \sin^2\psi)$$

$$= mg(\cos^2\psi - \sin^2\psi + \sin^2\psi_0)\sec\psi$$

$$= mg(\cos 2\psi + \sin^2\psi_0)\sec\psi$$. which is the reaction at any pt. on the curve. Oscillation continues backwards & forwards. ~~In passing through~~

Motion on a Rough Curve



Let a particle of mass m slide down a rough curve whose co-efficient of friction is μ. At any time t, P be the position of the particle. Distance of P from the fixed pt $A = s$.

R be the normal reaction at P

Eqn of motion $\quad mv\frac{dv}{ds} = \mu R - mg\sin\psi \quad \text{--- (1)}$

$$\frac{mv^2}{\rho} = R - mg\cos\psi \quad \text{--- (2)}$$

Eliminating R between (1) & (2), we have

$$v\frac{dv}{ds} - \frac{\mu v^2}{\rho} = g(\mu\cos\psi - \sin\psi)$$

or, $\frac{1}{2}\frac{dv^2}{ds} - \frac{\mu v^2}{\rho} = g(\mu\cos\psi - \sin\psi)$

or, $\frac{1}{2}\cdot\frac{dv^2}{d\psi}\cdot\frac{d\psi}{ds} - \frac{\mu v^2}{\rho} = g(\mu\cos\psi - \sin\psi)$

or, $\frac{1}{2\rho}\left[\frac{dv^2}{d\psi} - 2\mu v^2\right] = g(\mu\cos\psi - \sin\psi)$

or, $\frac{dv^2}{d\psi} - 2\mu v^2 = 2\rho g(\mu\cos\psi - \sin\psi) \quad \text{--- (3)}$.

Int. factor $e^{-\int 2\mu d\psi} = e^{-2\mu\psi}$

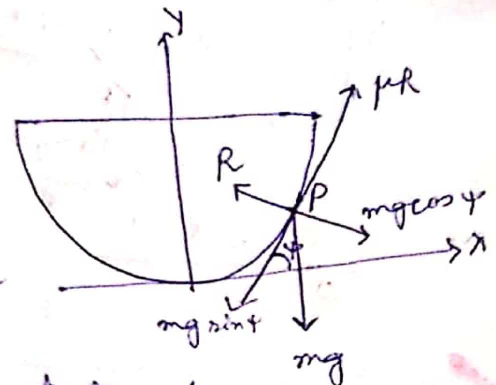
$\therefore$ Multiplying both sides of (3) by $e^{-2\mu\psi}$ and integrating

$$v^2 e^{-2\mu\psi} = \int 2\rho g(\mu\cos\psi - \sin\psi)e^{-2\mu\psi}\,d\psi + C \quad \text{--- (4)}$$

For known curve ρ is known in terms of ψ.

$\therefore$ On integration v^2, & from (2) R are obtained.

Motion in a Rough Cycloid under Gravity



Let a heavy particle of mass m slide inside a rough cycloid with its vertex at a lowest point and axis vertical.
Let μ be the coefficient of friction on the curve.

P be the position of the particle at time t. Forces acting on the particle are shown in the fig.

Eq^n of motions

$$mv\frac{dv}{ds} = \mu R - mg\sin\psi \quad \cdots (1)$$

$$\frac{mv^2}{\rho} = R - mg\cos\psi \quad \cdots (2)$$

Eliminating R between (1) & (2), we have

$$v\frac{dv}{ds} - \frac{\mu v^2}{\rho} = g(\mu\cos\psi - \sin\psi)$$

$$\text{or } \frac{1}{2}\frac{dv^2}{d\psi}\cdot\frac{d\psi}{ds} - \mu v^2\frac{d\psi}{ds} = g(\mu\cos\psi - \sin\psi)$$

$$\text{or } \frac{dv^2}{d\psi} - 2\mu v^2 = 2g(\mu\cos\psi - \sin\psi)\frac{ds}{d\psi} \quad \cdots (3)$$

$$s = 4a\sin\psi \quad [\text{Intrinsic eq}^n \text{ of cycloid}] \quad \cdots (4)$$

$$\frac{ds}{d\psi} = 4a\cos\psi$$

$$\therefore \frac{dv^2}{d\psi} - 2\mu v^2 = 8ag(\mu\cos\psi - \sin\psi)\cos\psi.$$

$$\text{or } \qquad = 4ag[\mu(1+\cos2\psi) - \sin2\psi] \quad \cdots (5)$$

I.F $e^{-2\mu\psi}$

$$\therefore \frac{d}{d\psi}(v^2e^{-2\mu\psi}) = 4ag\,e^{-2\mu\psi}[\mu(1+\cos2\psi) - \sin2\psi]$$

Integrating

$$v^2e^{-2\mu\psi} = 4ag\int e^{-2\mu\psi}\{\mu(1+\cos2\psi) - \sin2\psi\}\,d\psi$$

If $v = 0$ at $\psi = \psi_0$ i.e, the particle starts from rest from a pt where the tangent makes an angle ψ_0 with the horizontal, then

$$v^2e^{-2\mu\psi} = 4ag\int_{\psi_0}^{\psi} e^{-2\mu\psi}[\mu + \mu\cos2\psi - \sin2\psi]\,d\psi.$$

Do yourself.

$$= 4ag\left[-\frac{e^{-2\mu\psi}}{2} + \mu e^{-2\mu\psi}\left\{\frac{-2\mu\cos 2\psi + 2\sin 2\psi}{4(1+\mu^2)}\right\} - e^{-2\mu\psi}\left\{\frac{-2\mu\sin 2\psi - 2\cos 2\psi}{4(1+\mu^2)}\right\}\right]_{\psi_0}^{\psi}$$

Simplifying

$$= 4ag\left[-\frac{e^{-2\mu\psi}}{2} + \frac{e^{-2\mu\psi}}{2(1+\mu^2)}\left\{(1-\mu^2)\cos 2\psi + 2\mu\sin 2\psi\right\}\right]_{\psi_0}^{\psi}$$

$$= \frac{2ag}{1+\mu^2}\left[-e^{-2\mu\psi}\left\{(1+\mu^2) - (1-\mu^2)\cos 2\psi - 2\mu\sin 2\psi\right\}\right]_{\psi_0}^{\psi}$$

$$= \frac{2ag}{1+\mu^2}\left[e^{-2\mu\psi}\left\{-\mu^2(1+\cos 2\psi) - (1-\cos 2\psi) + 2\mu\sin 2\psi\right\}\right]_{\psi_0}^{\psi}$$

$$= \frac{2ag}{1+\mu^2}\left[e^{-2\mu\psi}\left\{-2\mu^2\cos^2\psi - 2\sin^2\psi + 4\mu\sin\psi\cos\psi\right\}\right]_{\psi_0}^{\psi}$$

$$= \frac{2ag}{1+\mu^2}\cdot 2\left[e^{-2\mu\psi}(\sin\psi - \mu\cos\psi)^2\right]_{\psi_0}^{\psi}$$

$$= \frac{4ag}{1+\mu^2}\left[e^{-2\mu\psi_0}(\sin\psi_0 - \mu\cos\psi_0)^2 - e^{-2\mu\psi}(\sin\psi - \mu\cos\psi)^2\right] \quad \text{--- (6)}$$

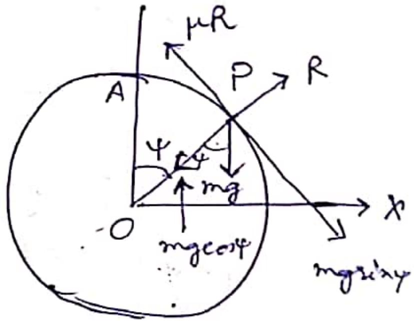
$\therefore$ using (6) R can be obtained from (2).

$$v = \frac{ds}{dt} = 4a\cos\psi\,\frac{d\psi}{dt}.$$

$$\int_0^t dt = \int_{\psi_0}^{\psi}\frac{4a\cos\psi}{v}\,d\psi$$

v is a f^n of ψ.
we can integrate for the given problem.

Motion on a Rough Circle under Gravity.



P be the position of the particle at any time t.

$\widehat{AP} = s$. $\quad s = a\psi$

R is the normal reaction

μR = tangential frictional force acting opposite to the motion.

Eqns of motion :-

$$mv\frac{dv}{ds} = mg\sin\psi - \mu R \quad \text{--- (1)} \qquad s = a\psi, \quad \frac{ds}{d\psi} = a.$$

$$\frac{mv^2}{\rho} = R - mg\cos\psi \quad \text{--- (2)}$$

Eliminating R from the above equations

$$mv\frac{dv}{d\psi}\cdot\frac{d\psi}{ds} + \frac{mv^2\mu}{\rho} = mg\sin\psi - mg\mu\cos\psi$$

$$2v\frac{dv}{d\psi} + 2\mu v^2 = 2ga(\sin\psi - \mu\cos\psi)$$

$$\frac{dv^2}{d\psi} + 2\mu v^2 = 2ga(\sin\psi - \mu\cos\psi)$$

Int. fact.

$$\frac{d}{d\psi}\left(v^2 e^{2\mu\psi}\right) = 2ga\, e^{2\mu\psi}(\sin\psi - \mu\cos\psi)$$

Int

$$v^2 e^{2\mu\psi} = 2ga\, e^{2\mu\psi}\frac{2\mu\sin\psi - \cos\psi - 2\mu^2\cos\psi - \mu\sin\psi}{1+4\mu^2} + C$$

$$v^2 = \frac{2ga}{1+4\mu^2}\left(\mu\sin\psi - (1+2\mu^2)\cos\psi\right) + Ce^{-2\mu\psi} \quad \text{--- (3)}$$

$$\left[\ \because \int e^{ax}\sin bx\,dx = e^{ax}\frac{a\sin bx - b\cos bx}{a^2+b^2}\right.$$

$$\left.\int e^{ax}\cos bx\,dx = e^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2}\ \right].$$

at $\psi = 0$, $\quad v = v_0$

$$v_0^2 = C - \frac{2ag}{1+4\mu^2}(1+2\mu^2).$$

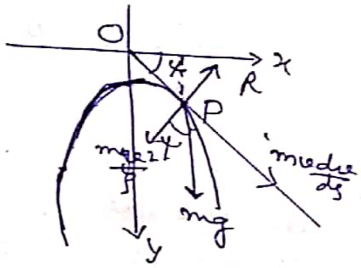
$$\therefore C = v_0^2 + \frac{2ag}{1+4\mu^2}(1+2\mu^2). \quad \text{--- (4)}$$

Substituting in (4) in (3), v^2 is obtained,

& from (2) using v^2 ... R can be obtained.

Ex A particle slides down the smooth curve $y = a\sinh\frac{x}{a}$, the axis of x being horizontal and the axis of y downwards, starting from rest at the point where the tangent is inclined at α to the horizon; shew that it will leave the curve when it has fallen through a vertical distance $a\sec\alpha$.

$$y = a\sinh\frac{x}{a}.$$



P be the position of the particle at time t.

$R \to$ normal reaction.

Eq^n of motion -

$$mv\frac{dv}{ds} = mg\sin\psi \quad \text{--- (1)}$$

$$\frac{mv^2}{\rho} = mg\cos\psi - R \quad \text{--- (2)}.$$

$y = a\sinh x/a$

$\frac{dy}{dx} = a\cdot\frac{1}{a}\cosh x/a$

$y_1 = \tan\psi = \cosh x/a.$

$= \sqrt{1+\sinh^2 x/a}$

$= \sqrt{1+\frac{y^2}{a^2}}$

when $\psi = \alpha$

$y = b$..

$$\tan\alpha = \sqrt{1+\frac{b^2}{a^2}}$$

$\frac{d^2y}{dx^2} = \frac{a}{a}\frac{1}{a}\sinh x/a = y/a^2.$

$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\cosh^2 x/a)^{3/2}}{y/a}$

$= \frac{\left(1+1+\frac{y^2}{a^2}\right)^{3/2}}{y/a^2}$

$= \frac{(2a^2+y^2)^{3/2}}{\frac{y}{a^2}\cdot a^3} = \frac{(2a^2+y^2)^{3/2}}{ay}$

from (1)

$$v\frac{dv}{ds} = g\sin\psi$$

$$= g\cdot\frac{dy}{ds}$$

or $2v\,dv = 2g\,dy$

Int $v^2 = 2gy + c$

$v = 0$, at $y = b$, $c = -2gb$

$$v^2 = 2g(y-b). \quad \text{--- (3)}.$$

The particle will leave the plane when $R = 0$.

$\Rightarrow \quad v^2 = g\rho\cos\psi$

or $2g(y-b) = g\rho\cos\psi = \frac{g\rho}{\sqrt{1+\tan^2\psi}}$

or, or $2(y-b) = \frac{\cancel{g}(2a^2+y^2)^{3/2}}{ay\sqrt{\left(1+1+\frac{y^2}{a^2}\right)}} = \frac{\cancel{g}(2a^2+y^2)^{3/2}}{ay\frac{\sqrt{2a^2+y^2}}{a}} = \cancel{g}\frac{(2a^2+y^2)}{y}$

or, $2(y-b) = \cancel{g}\cdot\frac{2a^2}{y} + y$

or, $y - 2b = \frac{2a^2\cancel{g}}{y}$, or

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$\Rightarrow \quad y^2 - 2by = 2a^2.$ $\qquad\qquad \tan^2\alpha = 1 + \frac{b^2}{a^2}$

$\Rightarrow \quad y^2 - 2by + b^2 = 2a^2 + b^2$ $\qquad\qquad a^2\tan^2\alpha = a^2 + b^2$

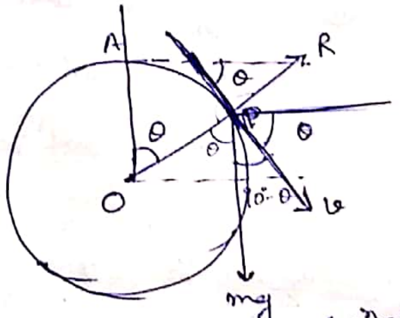
$\Rightarrow \quad (y-b)^2 = a^2 + a^2\tan^2\alpha$

$\qquad\qquad = a^2\sec^2\alpha$

$\therefore \quad y - b = a\sec\alpha$, proved.

Ex A particle starts from rest at the highest point on the outside of a smooth circle. Show that the latus rectum of the parabola, which the particle describes after leaving the circle is $\frac{16}{27}$ times the radius of the circle.

Ghosh Chakr. (4)



O – centre of the circle

P → position of the particle at any time t.

$\angle AOP = \theta$.

$\overset{\frown}{AP} = s$

v – vel. of the particle,

R → normal reaction

eqn of motion $\qquad\qquad s = a\theta$

$$m v\frac{dv}{ds} = mg\sin\theta \quad \text{--- (1)} \qquad \frac{ds}{d\theta} = a.$$

$$\frac{mv^2}{\rho} = mg\cos\theta - R \quad \text{--- (2)}$$

from (1) $\quad v\dfrac{dv}{a\,d\theta} = g\sin\theta$

or $\quad v\,dv = ag\sin\theta\,d\theta$

Int. $\quad v^2 = -2ag\cos\theta + c$

Initially $\quad t = 0,\ \theta = 0,\ v = 0 \quad \therefore\ c = 2ag$

$$v^2 = 2ag(1-\cos\theta) \quad \text{--- (3)}$$

from eqn (2) $\quad R = mg\cos\theta - \dfrac{mv^2}{a}$

$$= mg\cos\theta - \frac{2mg\cancel{a}}{\cancel{a}}(1-\cos\theta)$$

$$= mg(3\cos\theta - 2) \quad \text{--- (4)}$$

$R = 0$ → the particle leaves the circle, when $\theta = \alpha$

$3\cos\alpha - 2 = 0, \quad \cos\alpha = \frac{2}{3}$

$$v^2 = 2ga\left(1 - \frac{2}{3}\right) = \frac{2}{3}ag$$

$$v = \sqrt{\frac{2ag}{3}}.$$

horizontal component is $v\cos\alpha$

$$= v\sqrt{\frac{2}{3}ag}\cos\alpha \quad . \quad \cos\alpha = \frac{2}{3}.$$

$$= \frac{2}{3}\sqrt{\frac{2}{3}ag}.$$

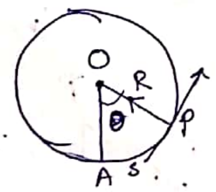
After leaving the circle, the particle describes parabolic path with vel. $\frac{2}{3}\sqrt{\frac{2}{3}ag}$.

Latus rectum of the parabola $= \frac{2}{g}\times(\text{hor. comp. of vel})^2$

$$= \frac{2}{g}\cdot\frac{4}{9}\cdot\frac{2ag}{3} = \frac{16a}{27}. \quad \text{proved.}$$

Ghosh & Chakraborty

(18) A particle is projected along the inner surface of a rough sphere and is acted upon by no forces. Show that it will return to the point of projection after time $\frac{a}{\mu V}(e^{2\pi\mu}-1)$, where a is the radius of the sphere, μ is the coefficient of friction and V is the velocity of projection.



The particle is projected from the lowest pt A ~~at the~~. P be the position at time t.

$\widehat{AP} = s$. [no forces $\Rightarrow$ no (mg)]

eq^n of motion

$$mv\frac{dv}{ds} = -\mu R \quad \text{--(1)}$$

$$\frac{mv^2}{\rho} = R \quad \text{--(2)}.$$

$$\therefore \cancel{m}v\frac{dv}{ds} = -\frac{\mu \cancel{m}v^2}{\rho} \qquad s = a\theta, \quad \frac{ds}{d\theta} = a = \rho.$$

or $\frac{dv^2}{ds} + \frac{2\mu v^2}{\rho} = 0$.

or $\frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds} + \frac{2\mu}{a}v^2 = 0$

or $\frac{dv^2}{d\theta} + 2\mu v^2 = 0$.

$$\frac{d}{d\theta}\left(v^2 e^{2\mu\theta}\right) = 0$$

Int $v^2 e^{2\mu\theta} = C$

Initially $v = V$, $\theta = 0$, $C = V^2$

$\therefore v^2 e^{2\mu\theta} = V^2$

$\therefore v e^{\mu\theta} = V$.

or

or. $\frac{ds}{dt} e^{\mu\theta} = V$

& $\frac{ds}{d\theta} \cdot \frac{d\theta}{dt} \cdot e^{\mu\theta} = V$ $\qquad \theta = 0$ when $t = 0$

$\theta = 2\pi$, $t = t_1$

$a e^{\mu\theta} d\theta = V dt$

& Int. $\int_0^{t_1} dt = \frac{a}{V}\int_0^{2\pi} e^{\mu\theta} d\theta$

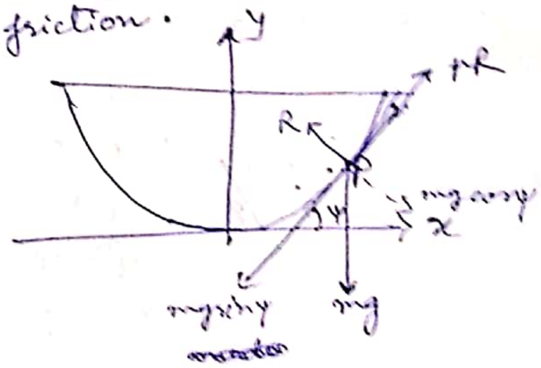
$t_1 = \frac{a}{\mu V}[e^{\mu\theta}]_0^{2\pi}$

$= \frac{a}{\mu V}[e^{2\mu\pi} - 1]$. proved.

Ghosh Chakr. (22)

Saha Ganguly (33)

A particle slides from a cusp down the arc of a rough cycloid, the axis of which is vertical. Prove that its velocity at the ~~point~~ vertex will bear to the velocity at the same point, when the cycloid is smooth, the ratio of $(e^{-\mu\pi} - \mu^2)^{\frac{1}{2}} : (1+\mu^2)^{\frac{1}{2}}$, where μ is the coefficient of friction.



P be the position of the particle

Eqn of motion

$mv\frac{dv}{ds} = \mu R - mg\sin\psi$

$\frac{mv^2}{\rho} = R - mg\cos\psi$

$\frac{1}{2}\frac{dv^2}{ds} - \frac{\mu v^2}{\rho} = g(\mu\cos\psi - \sin\psi)$

& $\frac{dv^2}{d\psi} \cdot \frac{d\psi}{ds} - \frac{2\mu v^2}{\rho} = 2g(\mu\cos\psi - \sin\psi)$

& $\frac{dv^2}{d\psi} - 2\mu v^2 = 2\rho g(\mu\cos\psi - \sin\psi)$

& $\frac{d}{d\psi}[v^2 e^{-2\mu\psi}] = 2g\int \rho e^{-2\mu\psi}(\mu\cos\psi - \sin\psi) d\psi$

I.F. $e^{-2\mu\psi}$, $s = 4a\sin\psi$

$\frac{ds}{d\psi} = 4a\cos\psi$.

Int $v^2 e^{-2\mu\psi} = 8ag\int e^{-2\mu\psi}(\mu\cos\psi - \sin\psi)\cos\psi\, d\psi + A$.

Let $z = e^{-\mu\psi}(\mu\cos\psi - \sin\psi)$

$dz = [e^{-\mu\psi}(-\mu\sin\psi - \cos\psi) - \mu e^{-\mu\psi}(\mu\cos\psi - \sin\psi)] d\psi$

$= -e^{-\mu\psi}\cos\psi(1+\mu^2) d\psi$

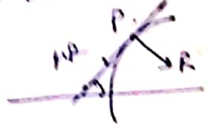
Bhattacharyya

H.W

1\. A heavy bead slides on a smooth fixed circular wire of radius a. It is projected from the lowest point with a velocity just sufficient to carry it to the highest pt. Prove that in time t the radius through the bead will turn through an angle $2\tan^{-1}\left(\sinh\sqrt{\frac{g}{a}}\,t\right)$ and that the bead will take an infinite time to reach the highest point.

[hint : same as first ex done yesterday (26.3.2020).

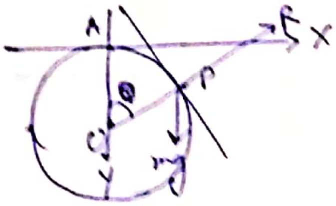
2\. A bead moves along a rough curved wire which is such that it changes its direction of motion with constant angular velocity. Such that a possible form of the wire is an equiangular spiral.

[hint :  $mv\frac{dv}{ds} = -\mu R$ $\quad \frac{d\psi}{dt} = \omega$ const]

$\frac{mv^2}{S} = R$

eq^n of equiangular spiral $s = Ae^{\mu\psi}$

3\. A particle starts from rest and slides from the highest point of a smooth circle. Prove that it will leave the curve when it has fallen a vertical height of one-third the radius of the circle and its velocity is then is $\sqrt{\frac{2ag}{3}}$.

hint :- 

Submit on 27.3.2020
8-30 a.m.

If v_1 be the velocity at the lowest point ($y = 0$), then

$$\int_0^{v_1} v\,dv = -g\int_{20}^{0} dy, \quad \text{or,} \quad \frac{1}{2}v_1^2 = -g(0-20) = 20g$$

$$\text{or,} \quad v_1^2 = 40g; \quad \therefore \quad v_1 = \sqrt{40g} = 2\sqrt{10g}.$$

Normal equation of motion is $m\frac{v^2}{\rho} = R - mg\cos\psi$, where R is the normal pressure.

$$\text{or,} \quad R = m\left(\frac{v^2}{\rho} + g\cos\psi\right)$$

$$\text{and} \quad \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{4x^2}{25}\right)^{3/2}}{2/5} = \frac{5}{2}\cdot\frac{(25+4x^2)^{3/2}}{125}.$$

From (2),

$$v^2 = 2g(20-y) = 2g\left(20 - \frac{1}{5}x^2\right) = \frac{2g}{5}(100-x^2).$$

$$\therefore \quad R = m\cdot\left\{\frac{2g}{5}(100-x^2)\times\frac{50}{(25+4x^2)^{3/2}} + g\cdot\frac{5}{\sqrt{4x^2+25}}\right\}$$

$$= 5mg\left\{\frac{400-4x^2+25+4x^2}{(25+4x^2)^{3/2}}\right\} = 5mg\cdot\frac{425}{(25+4x^2)^{3/2}}.$$

At $x = 5$,

$$R = 5mg\cdot\frac{425}{(25+100)^{3/2}} = 5mg\cdot\frac{425}{125(5)^{3/2}} = \frac{17}{5\sqrt{5}}mg.$$

❑ **Example 11.** *A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc; show that the time of reaching the vertex is*

$$2\sqrt{\frac{a}{g}}\tan^{-1}\left[\sqrt{\frac{4ag}{V}}\right].$$

[C.U. B.A./B.Sc.(H)'79]

Solution. Let P be the position of the particle at time t on the smooth cycloid AOB, where s is the arcual distance of P from the vertex O.

Let R be the normal reaction at P.

The equations of motion are

$$mv\frac{dv}{ds} = -mg\sin\psi \qquad (1)$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi, \qquad (2)$$

where mg is the weight of the particle acting vertically downwards.

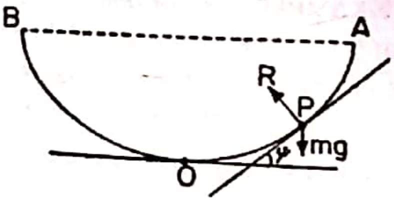


Fig. 7.20

Now, the equation of the curve is

$$s = 4a\sin\psi, \quad \text{or,} \quad \sin\psi = s/4a;$$

$\therefore$ From (1), $v\dfrac{dv}{ds} = -\dfrac{gs}{4a}$ and integrating, $v^2 = -\dfrac{g}{4a}.s^2 + C$, where C is the constant of integration.

Initially, when $s = 4a, v = -V$; $\therefore$ $C = V^2 + 4ag$.

Hence, $v^2 = \dfrac{g}{4a}(16a^2 - s^2) + V^2$;

$$\therefore \quad \frac{ds}{dt} = v = -\sqrt{\frac{g}{4a}(16a^2 - s^2) + V^2},$$

[the negative sign is taken, since the particle is moving down the curve, i.e., in the direction of s decreasing.] .

$$\text{or,} \quad \sqrt{\frac{g}{4a}}\,dt = -\frac{ds}{\sqrt{16a^2 + \frac{4aV^2}{g} - s^2}}$$

Integrating,

$$\frac{1}{2}\sqrt{\frac{g}{a}}\int_0^T dt = -\int_{4a}^{0}\frac{ds}{\sqrt{16a^2 + \frac{4aV^2}{g} - s^2}}$$

[since, when $t = 0, s = 4a$ and when $t = T$ (say), $s = 0$, i.e., the particle reaches the vertex 0 at time T.]

$$\therefore \quad \frac{1}{2}\sqrt{\frac{g}{a}}T = \left[\cos^{-1}\frac{s}{\sqrt{16a^2 + \frac{4aV^2}{g}}}\right]_{4a}^{0} = \frac{\pi}{2} - \cos^{-1}\frac{4a}{\sqrt{16a^2 + \frac{4aV^2}{g}}}$$

$$= \sin^{-1}\frac{4a}{\sqrt{16a^2 + \frac{4aV^2}{g}}} = \sin^{-1}\frac{1}{\sqrt{1 + \frac{V^2}{4ag}}} = \tan^{-1}\frac{\sqrt{4ag}}{V},$$

$$\text{or,} \quad T = 2\sqrt{\frac{a}{g}}\tan^{-1}\frac{\sqrt{4ag}}{V}.$$

❑ EXAMPLE 12. *A cycloid is placed with its axis vertical and vertex upwards and a heavy particle is projected from the cusp up the concave side of the curve with velocity $\sqrt{2gh}$; prove that the latus rectum of a parabola described after leaving the arc is $\frac{h^2}{2a}$ where a is the radius of the generating circle.* [C.U. B.A./B.Sc.(H)'67,'69]

SOLUTION. Let the particle be projected from the cusp A where $s = 4a$ and let P be the position of the particle at any time t, where arc $OP = s$.

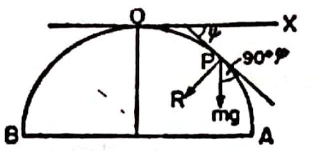


Fig. 7.21

Then the equations of motion are

$$mv\frac{dv}{ds} = mg\sin\psi = mg\cdot\frac{s}{4a}$$
$$[\because\ s = 4a\sin\psi] \quad (1)$$

$$\text{and} \quad m\frac{v^2}{\rho} = R + mg\cos\psi \quad (2)$$

From (1),

$$v\frac{dv}{ds} = \frac{gs}{4a}, \quad \text{or,} \quad v\,dv = \frac{g}{4a}s\,ds.$$

Integrating, $v^2 = \frac{gs^2}{4a} + A_1$, where A_1 is the constant of integration.
Initially at $A, v = \sqrt{2gh}, s = 4a$; $\therefore\ 2gh = 4ag + A_1$.

$$\therefore\ v^2 = \frac{gs^2}{4a} + 2gh - 4ag.$$

Putting $s = 4a\sin\psi$,

$$v^2 = 2gh + 4ag\sin^2\psi - 4ag$$
$$\text{or,}\quad v^2 = 2gh - 4ag\cos^2\psi. \quad (3)$$

From (2), using (3), we get

$$R = \frac{m(2gh - 4ag\cos^2\psi)}{4a\cos\psi} - mg\cos\psi \quad \left[\because\ \rho = \frac{ds}{d\psi} = 4a\cos\psi\right]$$
$$= \frac{mg}{4a\cos\psi}[2h - 8a\cos^2\psi].$$

The particle will leave the cycloid, when $R = 0$, i.e., when $2h - 8a\cos^2\psi = 0$, or, $\cos^2\psi = \frac{h}{4a}$, which gives the value of ψ when it leaves the arc.

If u be the velocity with which the particle leaves the arc, then from (3), using $\cos^2\psi = \frac{h}{4a}$, we get

$$u^2 = 2gh - 4ag\cdot\frac{h}{4a} = gh.$$

Now, we know that the latus rectum of the parabolic path is $\frac{2u^2\cos^2\alpha}{g}$, when a particle is projected with velocity u and at an angle α.

Hence the latus rectum of the parabola described after leaving the arc

$$= \frac{2u^2\cos^2\alpha}{g} = \frac{2u^2\cos^2\psi}{g} = \frac{2gh\cdot\frac{h}{4a}}{g} = \frac{\mathbf{h^2}}{\mathbf{2a}}.$$

❑ EXAMPLE 13. *A bead slides on a smooth parabolic wire held in a vertical plane with axis vertical and vertex downwards starting from rest at a height h above the vertex. Show that the square of the normal reaction at any point is inversely proportional to the cube of the height of the point above the directrix.*

[C.U. B.A./B.Sc.(H)'52,'64]

SOLUTION. Let $P(x,y)$ be the position of the bead of mass m at any time t with reference to the vertex O as origin, the tangent Ox at O as the x-axis and the upward vertical through O as the y-axis.

Then the equation of the parabola is

$$x^2 = 4ay. \quad (1)$$

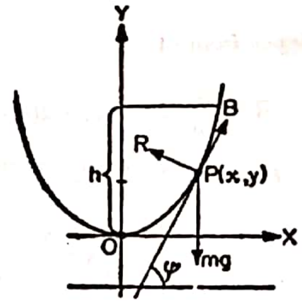


Fig. 7.22

The equations of motion are

$$mv\frac{dv}{ds} = -mg\cos(90^0 - \psi) \quad (2)$$

$$\text{and}\quad m\cdot\frac{v^2}{\rho} = R - mg\sin(90^0 - \psi). \quad (3)$$

From (2),

$$v\frac{dv}{ds} = -g\sin\psi = -g\frac{dy}{ds}, \quad \text{or,}\quad 2v\,dv = -2g\,dy;$$
$$\therefore\ v^2 = -2gy + A,$$

where A is the constant of integration.

Initially, at $y = h, v = 0$; $\therefore\ O = -2gh + A$, or, $A = 2gh$.

$$\therefore\ v^2 = 2g(h - y).$$

From (3),

$$R = \frac{mv^2}{\rho} + mg\cos\psi = \frac{2mg(h-y)}{\rho} + mg\cos\psi.$$
$$= \frac{mg}{\rho}[2(h-y) + 2\rho\cos\psi]. \quad (4)$$

Now, from (1),

$$\frac{dy}{dx} = \frac{x}{2a} = \tan\psi \quad (5)$$
$$\therefore\ x = 2a\tan\psi$$

$$\text{or,}\quad \frac{dx}{d\psi} = 2a\sec^2\psi = 2a\left(1 + \frac{x^2}{4a^2}\right) = 2a\left(1 + \frac{4ay}{4a^2}\right)$$

[from (5) and (1)]

$$\text{or,}\quad \frac{dx}{ds}\cdot\frac{ds}{d\psi} = 2a\left(1 + \frac{y}{a}\right) = 2(a+y)$$

$$\text{or,}\quad \rho\cos\psi = 2(a+y). \quad (6)$$

or, $2v \cdot \dfrac{dv}{d\psi} - 2\mu v^2 = 2g(\mu \cos\psi - \sin\psi) 4a\cos\psi$

[Since for a cycloidal curve, $s = 4a \sin\psi$]

or, $\dfrac{d(v^2)}{d\psi} - 2\mu \cdot v^2 = 4ag(\mu \cdot 2\cos^2\psi - 2\sin\psi\cos\psi)$

or, $\dfrac{d(v^2)}{d\psi} - 2\mu \cdot v^2 = 4ag\{\mu(1+\cos 2\psi) - \sin 2\psi\}$. (3)

This is a *linear equation in* v^2 *and* ψ, whose integrating factor is

$$e^{\int -2\mu d\psi}, \quad \text{i.e.,} \quad e^{-2\mu\psi}$$

Hence, the solution of (3) is

$$v^2 \cdot e^{-2\mu\psi} = 4ag \int e^{-2\mu\psi}\{\mu(1+\cos 2\psi) - \sin 2\psi\}\, d\psi + C,$$

where C is the constant of integration,

$$\text{or,}\quad v^2 e^{-2\mu\psi} = 4ag\left[-\frac{e^{-2\mu\psi}}{2} + \frac{\mu e^{-2\mu\psi}}{4(1+\mu^2)}(-2\mu\cos 2\psi + 2\sin 2\psi) - \frac{e^{-2\mu\psi}}{4(1+\mu^2)}(-2\mu \sin 2\psi - 2\cos 2\psi)\right] + C$$

$$= \frac{age^{-2\mu\psi}}{1+\mu^2}\left[-2(1+\mu^2) + \mu(-2\mu\cos 2\psi + 2\sin 2\psi) + (2\mu\sin 2\psi + 2\cos 2\psi)\right] + C.$$

$$= \frac{2ag \cdot e^{-2\mu\psi}}{1+\mu^2}[-(1+\mu^2) + (1-\mu^2)\cos 2\psi + 2\mu \sin 2\psi] + C.$$

$$= -\frac{2ag \cdot e^{-2\mu\psi}}{1+\mu^2}[2\sin^2\psi + 2\mu^2\cos^2\psi - 4\mu\sin\psi\cos\psi] + C$$

$$= -\frac{4ag \cdot e^{-2\mu\psi}}{1+\mu^2}(\sin\psi - \mu\cos\psi)^2 + C. \quad (4)$$

Now, since $v = 0$ when $\psi = 0$ and $\frac{\pi}{2}$, from (4) we have

$$0 = -\frac{4ag}{1+\mu^2} \cdot \mu^2 + C \quad (5)$$

$$\text{and}\quad 0 = -\frac{4ag \cdot e^{-\mu\pi}}{1+\mu^2} + C. \quad (6)$$

From (5) and (6), we get $\mu^2 = e^{-\mu t}$, or, $\mu^2 e^{\mu\pi} = 1$.

❑ **EXAMPLE 16.** *A particle is projected along the inner surface of a rough sphere and is acted on by no forces. Show that it will return to the point of projection at the end of time $\frac{a}{\mu V}(e^{2\pi\mu} - 1)$, where a is the radius of the sphere, V is the velocity of projection and μ is the coefficient of friction.* [B.U.(H)'82;C.U.(H)'78,'81;V.U.(H)'91]

SOLUTION. Since the particle is acted on by no forces, it will continue to move in a circle (circular section of the sphere).

Let the particle be projected from the point A with velocity V and P be its position at any time t such that

$$\angle AOP = \theta \quad \text{and arc} \quad AP = s.$$

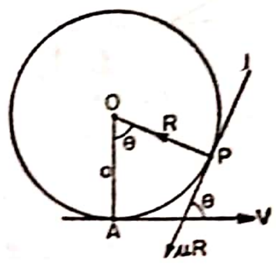


Fig. 7.25

Then $s = a\theta$, $\dot{s} = a\dot{\theta}$ and $\ddot{s} = a\ddot{\theta}$.

If R be the normal reaction at P, then μR is the force of friction acting opposite to the direction of motion. The equations of motion are

$$\ddot{s} = -\mu R \quad \text{and} \quad \frac{v^2}{\rho} = R,$$

or, $a\ddot{\theta} = -\mu R$ (1)

and $a\dot{\theta}^2 = R$ [$\because$ $\rho = a$] (2)

From (1) and (2), eliminating R, we get,

$$a\ddot{\theta} = -\mu a\dot{\theta}^2, \quad \text{or,} \quad \ddot{\theta} + \mu\dot{\theta}^2 = 0, \quad \text{or,} \quad \frac{\ddot{\theta}}{\dot{\theta}} + \mu\dot{\theta} = 0.$$

Integrating, $\log\dot{\theta} + \mu\theta = C$, where C is a constant.

Initially, at $A, \theta = 0, \dot{\theta} = \frac{\dot{s}}{a} = \frac{V}{a}$; $\therefore$ $\log\frac{V}{a} = C$.

$$\therefore \quad \log\dot{\theta} + \mu\theta = \log\frac{V}{a}, \quad \text{or,} \quad \log\left(\dot{\theta}/\frac{V}{a}\right) = -\mu\theta;$$

$$\therefore \quad \dot{\theta} = \frac{V}{a} \cdot e^{-\mu\theta}, \quad \text{or,} \quad dt = \frac{a}{V}e^{\mu\theta}\, d\theta.$$

The particle will return to the point of projection when $\theta = 2\pi$. If T be the required time, then

$$\int_0^T dt = \int_0^{2\pi} \frac{a}{V}e^{\mu\theta} \cdot d\theta, \quad \text{or,} \quad T = \frac{a}{V}\left[\frac{e^{\mu\theta}}{\mu}\right]_0^{2\pi}$$

Hence $\mathbf{T} = \frac{a}{\mu V}(e^{2\pi\mu} - 1)$.

❑ **EXAMPLE 17.** *A particle is projected horizontally from the lowest point of a rough sphere of radius a, and comes to rest at the lowest point after describing an arc less than a quadrant, show that the velocity of projection must be $\sin\alpha\sqrt{2ga(1+\mu^2)/(1-2\mu^2)}$, where α is the angle subtended by the arc at the centre and μ is the coefficient of friction.* [C.U. B.A./B.Sc.(H)'70,'85;(P)'85]

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Ex. 2. *A smooth parabolic tube is placed, vertex downwards, in a vertical plane. A particle slides down the tube from rest under the influence of gravity. Prove that, in any position, the reaction of the tube is* $\frac{2w(h+a)}{\rho}$, *where w is the weight of the particle,* ρ *is the radius of curvature, 4a is the latus rectum and h is the original height of the particle above the vertex.* [C. H. 1996]

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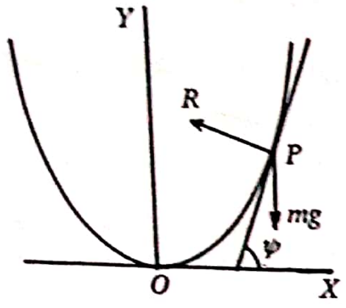

Fig. 13.8 (a)

Let the equation of the parabola with axis as the y-axis be $x^2 = 4ay$.

Therefore $\rho = \dfrac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

or, $\rho = 2a(1+\tan^2\psi)^{\frac{3}{2}}$, since $\frac{dy}{dx} = \tan\psi = \frac{x}{2a}$

$= 2a\sec^3\psi$

$$= \frac{2a}{\cos\psi}\left(1+\frac{x^2}{4a^2}\right) = \frac{2a}{\cos\psi}\left(1+\frac{y}{a}\right) = \frac{2(a+y)}{\cos\psi}. \quad \dots (1)$$

Equation of motion along the tangent is

$$mv\frac{dv}{ds} = -mg\sin\psi$$

or, $\quad v\frac{dv}{ds} = -g\frac{dy}{ds}$.

Integrating, we get $v^2 = C - 2gy$.

When $y = h$, we have $v = 0$.

Therefore $\quad C = 2gh$.

$$\dots (2)$$

Therefore $\quad v^2 = 2g(h-y)$.

Equation of motion along the normal is

$$m\frac{v^2}{\rho} = R - mg\cos\psi.$$

Therefore $R = m\left(g\cos\psi + \frac{v^2}{\rho}\right)$

$$= m\left\{g\cdot\frac{2(a+y)}{\rho} + \frac{1}{\rho}2g(h-y)\right\}, \text{ by (1) and (2)}$$

$$= 2mg\cdot\frac{h+a}{\rho} = \frac{2w(h+a)}{\rho}.$$

Ex. 3. *A particle falls down a cycloid under its own weight starting from the cusp. Show that when it arrives at the vertex the pressure on the curve is twice the weight of the particle.*

We have, as usual, the equations of motion

$$v\frac{dv}{ds} = -g\sin\psi = -\frac{g}{4a}s, \quad \dots (1)$$

$$m\frac{v^2}{\rho} = R - mg\cos\psi. \quad \dots (2)$$

Integrating equation (1), we get

$$v^2 = -\frac{g}{4a}s^2 + C. \quad \dots (3)$$

Initially, $\quad v = 0, \quad$ when $s = 4a$.

Therefore $\quad C = \frac{g}{4a}\cdot 16a^2$.

Hence (3) reduces to $v^2 = \frac{g}{4a}(16a^2 - s^2)$.

At the vertex, $s = 0$; hence $v^2 = 4ag$ at the vertex.

Hence, from (2), $R = \frac{mv^2}{\rho} + mg\cos\psi = m\cdot\frac{4ag}{4a} + mg = 2mg$,

since $s = 4a\sin\psi$ and $\rho = 4a\cos\psi$, so that, at $\psi = 0$, $\rho = 4a$.

Ex. 4. *A heavy particle hangs from a point O by a string of length a. It is projected horizontally with a velocity v such that* $v^2 = (2+\sqrt{3})ag$. *Show that the string becomes slack when it has described an angle*

$$\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right).$$

Let the particle be at P at time t such that $AP = s$ and $\angle AOP = \theta$, where A is the initial position of the particle. The forces acting on the particle are

(i) weight mg acting vertically downwards,

(ii) tension T acting along the string towards O.

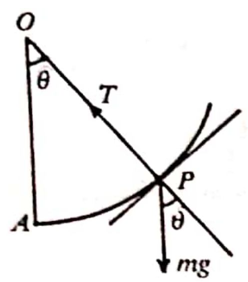

Fig. 13.8 (b)

The equations of motion are

$$mv\frac{dv}{ds} = -mg\sin\theta \qquad \ldots (1)$$

and

$$m\frac{v^2}{\rho} = T - mg\cos\theta. \qquad \ldots (2)$$

Now $s = a\theta$, so that $\dfrac{ds}{d\theta} = a$.

From (1), we have

$$v\frac{dv}{d\theta} = -ag\sin\theta.$$

Integrating, we get

$$v^2 = 2ag\cos\theta + C, \text{ where C is a constant.} \qquad \ldots (3)$$

Initially, at A, $\theta = 0$ and $v^2 = (2 + \sqrt{3})\,ag$.

Hence $C = \sqrt{3}ag$.

Substituting in (3), we get

$$v^2 = 2ag\cos\theta + \sqrt{3}ag. \qquad \ldots (4)$$

From (2) and (4), we have

$$\frac{m}{a}(2ag\cos\theta + \sqrt{3}ag) = T - mg\cos\theta$$

or, $$T = m(3g\cos\theta + \sqrt{3}g). \qquad \ldots (5)$$

The string becomes slack when $T = 0$.

Hence, from (5), we have

$$3g\cos\theta + \sqrt{3}g = 0$$

or, $$\cos\theta = -\frac{1}{\sqrt{3}}, \text{ giving } \theta = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\right).$$

Ex. 5. *A bead slides down a rough circular wire, which is in a vertical plane, starting from rest at the end of the horizontal diameter. When it has described an angle θ about the centre, show that the square of the angular velocity is*

$$\frac{2g}{a(1+4\mu^2)}\{(1-2\mu^2)\sin\theta + 3\mu(\cos\theta - e^{-2\mu\theta})\},$$

where μ is the coefficient of friction and a is the radius of the circle.

[B. H. 1992; C. H. 2001]

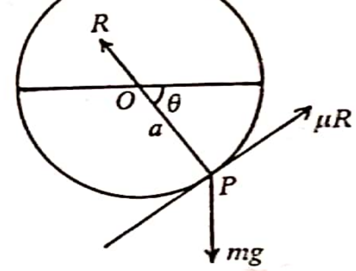

Fig. 13.8 (c)

The forces acting on the particle are

(i) mg vertically downwards,

(ii) reaction R normal to the path,

(iii) friction μR tangentially.

Taking the centre of the circle as pole and the horizontal diameter as initial line, the co-ordinates of P are (a, θ).

Equations of motion in the tangential and the normal directions are

$$ma\ddot{\theta} = mg\cos\theta - \mu R,$$

$$ma\dot{\theta}^2 = R - mg\sin\theta,$$

since $v = a\dot{\theta}$ and $\rho = a$, so that $\dfrac{dv}{dt} = a\ddot{\theta}$ and $\dfrac{v^2}{\rho} = a\dot{\theta}^2$.

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Eliminating R from the equations, we get

$$\ddot{\theta} + \mu\dot{\theta}^2 = \frac{g}{a}(\cos\theta - \mu\sin\theta)$$

or, $$\frac{d}{d\theta}(\dot{\theta}^2) + 2\mu\dot{\theta}^2 = \frac{2g}{a}(\cos\theta - \mu\sin\theta).$$

Integrating, we get

$$\dot{\theta}^2 e^{2\mu\theta} = \frac{2g}{a}\int e^{2\mu\theta}(\cos\theta - \mu\sin\theta)\,d\theta$$

$$= \frac{2g}{a}\left(\frac{2\mu e^{2\mu\theta}\cos\theta + e^{2\mu\theta}\sin\theta}{1+4\mu^2} - \mu\,\frac{2\mu\, e^{2\mu\theta}\sin\theta - e^{2\mu\theta}\cos\theta}{1+4\mu^2}\right) + C$$

$$= \frac{2g}{a}\,\frac{e^{2\mu\theta}}{1+4\mu^2}\{3\mu\cos\theta + (1-2\mu^2)\sin\theta\} + C.$$

Initially, $\theta = 0 = \dot{\theta}$. Hence $\frac{2g}{a}\cdot\frac{3\mu}{1+4\mu^2} + C = 0.$

Therefore $C = -\frac{2g}{a}\,\frac{3\mu}{1+4\mu^2}.$

Hence the result follows.

Ex. 6. *A heavy particle slides down a rough cycloid of which the coefficient of friction is* μ. *Its base is horizontal and vertex downwards. Show that*

(i) if it starts from rest at the cusp and comes to rest at the vertex, then

$$\mu^2 e^{\mu\pi} = 1;$$ [*C. H. 1986 ; B. H. 1990*]

(ii) if it starts from rest at a point where the tangent makes an angle θ *with the horizontal and comes to rest at the vertex, then*

$$\mu e^{\mu\theta} = \sin\theta - \mu\cos\theta.$$ [*B. H. 1984 ; C. H. 1984*]

The motion being down the cycloid, the friction μR acts up the tangent.

The equations of motion are then

$$mv\frac{dv}{ds} = \mu R - mg\sin\psi$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi.$$

Eliminating R, we get

$$v\frac{dv}{ds} - \mu\frac{v^2}{\rho} = g(\mu\cos\psi - \sin\psi)$$

or, $$\frac{1}{2}\frac{d}{ds}(v^2) - \mu\frac{v^2}{\rho} = g(\mu\cos\psi - \sin\psi)$$

or, $$\frac{dv^2}{d\psi} - 2\mu v^2 = 2g\rho(\mu\cos\psi - \sin\psi).$$

Integrating, we get

$$v^2 e^{-2\mu\psi} = 2g\int \rho e^{-2\mu\psi}(\mu\cos\psi - \sin\psi)\,d\psi.$$

Now, for a cycloid, $s = 4a\sin\psi$ and $\rho = 4a\cos\psi$.

Hence $$v^2 e^{-2\mu\psi} = 8ag\int e^{-2\mu\psi}\cos\psi(\mu\cos\psi - \sin\psi)\,d\psi.$$

Put $$z = e^{-\mu\psi}(\mu\cos\psi - \sin\psi).$$

Then $$dz = -e^{-\mu\psi}(1+\mu^2)\cos\psi\,d\psi.$$

Hence we have

$$v^2 e^{-2\mu\psi} = -\frac{8ag}{1+\mu^2}\int z\,dz = -\frac{4ag}{1+\mu^2}z^2 + A,$$

A being a constant

$$= -\frac{4ag}{1+\mu^2}(\mu\cos\psi - \sin\psi)^2 e^{-2\mu\psi} + A. \quad \ldots (1)$$

(*i*) In this case, the conditions are $v = 0$ when $\psi = \frac{1}{2}\pi$ and $\psi = 0$.

Putting these values in (1), we get

$$A = \frac{4ag}{1+\mu^2}e^{-\pi\mu} = \frac{4ag}{1+\mu^2}\mu^2.$$

Therefore $\mu^2 = e^{-\pi\mu}$, whence $\mu^2 e^{\pi\mu} = 1$.

(*ii*) In this case, the conditions are $v = 0$ when $\psi = \theta$ and $\psi = 0$. Putting these values in (1), we get

$$A = \frac{4ag}{1+\mu^2}(\mu\cos\theta - \sin\theta)^2 e^{-2\mu\theta} = \frac{4ag}{1+\mu^2}\mu^2.$$

Therefore $$\mu^2 e^{2\mu\theta} = (\sin\theta - \mu\cos\theta)^2$$

or, $$\mu e^{\mu\theta} = \sin\theta - \mu\cos\theta.$$