Theorem: If A and B are sets of real numbers, then i) $A \subset B \Rightarrow A' \subset B'$

ii) $(A \cup B)' = A' \cup B'$

iii) $(A \cap B)' \subset A' \cap B'$

Give an example to show that $(A \cap B)'$ and $A' \cap B'$ may not be equal.

Solution: Let $A \subset B$.

Let $x \in A'$ $\Rightarrow$ x is a limit pt of A.

$\Rightarrow$ for $\epsilon > 0$ $]x-\epsilon, x+\epsilon[$ contains a pt $x_1 \in A$ such that $x_1 \neq x$.

$\because$ $A \subset B \Rightarrow x_1 \in B$.

$\therefore$ $]x-\epsilon, x+\epsilon[$ contains a pt $x_1 \in B$, & $x_1 \neq x$.

$\Rightarrow$ x is a limit pt of B, $x \in A' \Rightarrow x \in B'$

$\therefore$ $x \in B'$

$\therefore$ $A' \subset B'$.

ii) $A \subset A \cup B$, similarly $B \subset A \cup B$

$\Rightarrow A' \subset (A \cup B)'$ $\Rightarrow B' \subset (A \cup B)'$

$\Rightarrow A' \cup B' \subset (A \cup B)'$.

Let $x \in (A \cup B)'$

$\Rightarrow$ x is a limit pt of $A \cup B$

$\Rightarrow$ every nbd of x contains a pt $x_1 \in A \cup B$, $x_1 \neq x$.

$\Rightarrow$ " " " " " ($x_1 \in A$, or $x_1 \in B$) $x_1 \neq x$

$\Rightarrow$ " " " x " " " $x_1 \in A$, $x_1 \neq x$

or " " " x " " " $x_1 \in B$.

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

$\therefore (A \cup B)' \subset A' \cup B'$

$\therefore (A \cup B)' = A' \cup B'$.

iii) $A \cap B \subset A$

$(A\cap B)' \subset A'$

$A\cap B \subset B$

$(A\cap B)' \subset B'$

$(A\cap B)' \subset A'\cap B'$

To prove equality does not hold

$A =]1, 2[$ & $B =]2,3[$

$A \cap B = \phi$

$A' = [1,2]$ $B' = [2,3]$

$A'\cap B' = [2]$

Th (a) If a non-empty sub-set of R which is bounded above has no maximum member then show that its supremum is a limit pt of the set S.

(b) If a non empty sub set of R which is bounded below has no minimum member then show that its infimum is a limit pt of the set S.

Proof : (a) Since S is bounded above, by order completeness property ~~it~~ S has a supremum in R

let $\sup S = M$, $M \notin S$

let $\epsilon > 0$, $\sup S = M$ $\therefore M - \epsilon$ can not be an upperbound of S.

$\therefore \exists\, x > M - \epsilon$.

$\sup S = M$, $M + \epsilon > b$, $M+\epsilon$ is also an upper bound of S. $M+\epsilon > x$

$M - \epsilon < x < M + \epsilon$, $M \notin S$. $x \neq M$.

$\forall\ \epsilon > 0$, nbd $]M-\epsilon, M+\epsilon[$ contains a pt x of S which is different from b.

$\therefore$ b is a limit pt of ~~S~~ S.

$\sup S =$ limit pt of S

(b) Similar proof

Th Show that the derived set of any bounded set is also a bounded set.

Sol^n

Proof: Let S be a bounded set
$\therefore$ there exist $m, M \in R$ $S \subset [m, M]$

We shall prove that no elements of S' is less than m or greater than M.

If $x > M$ then for $\epsilon = x - M > 0$

$]x-\epsilon, x+\epsilon[$ is a nbd of x containing no element of S $\therefore$ $x \notin S'$

Similarly if $y < m$.

$\delta = m - y > 0$

$]y - \delta, y + \delta[$ is a nbd of y containing no element of S. $\therefore$ $y \notin S'$

$\therefore \Rightarrow$ all the limit pts lie in $[m, M]$

$\Rightarrow$ $S' \subset [m, M]$

$\Rightarrow$ Derived set of a bounded set is bounded.

Th Every infinite bounded set has the greatest and the smallest limit points i.e, the derived set of an infinite bounded set attains its bounds.

OR

Let S be a bounded infinite non-empty subset of R. Then the derived set S has the smallest and the greatest members.

Solution Let S be an infinite bounded set.

Since S is bounded, so there exist $h, k \in R$ s.t $S \subset [h, k]$

$\Rightarrow S' \subset [h, k]'$, derived set of $[h,k] = [h,k]$

$\Rightarrow S'$ is bounded. $S' \subset [h,k]$

$\therefore$ S is an infinite bounded set.

$\therefore$ By Bolzano-Weierstrass th, S has a limit pt. $\therefore S' \neq \phi$

Hence S' is non empty and bounded and so by order completeness property of $\mathbb{R}$ ~~S has a limit~~ S' has both infimum and supremum.

Let $\inf S' = a$ and $\sup S' = b$.

We shall prove that both a and $b \in S'$.

We shall prove that both a and b are limit pts of S.

i.e. both a and b

$\because a = \inf S'$

$\Rightarrow$ for any $\epsilon > 0$, $\exists$ some $x \in S'$ $a \le x < a+\epsilon$

$a - \epsilon < a \le x < a + \epsilon$.

$]a-\epsilon, a+\epsilon[$ nbd of $x \in S'$

$\Rightarrow]a-\epsilon, a+\epsilon[$ is a limit pt of S

$\Rightarrow]a-\epsilon, a+\epsilon[$ is nbd of

$\Rightarrow]a-\epsilon, a+\epsilon[$ contains infinite pts of S.

$\Rightarrow$ ~~a is~~ every nbd of a contains infinitely many pts of S.

$\Rightarrow$ a is a limit pt of S.

Similarly b is a -- -- S

a is the smallest and b is the greatest ~~ti~~ member of S.

Observation

1. The greatest and least limit points of an infinite bounded set S are often called the upper limit μ and the lower limit λ.

2. μ and λ have the following characteristic properties:

i) For any $\epsilon > 0$, the set S has infinitely many elements $> \mu - \epsilon$ and has only finite number of elements $> \mu + \epsilon$.

ii) For any $\epsilon >$, the set has infinitely many elements $< \lambda + \epsilon$ and has only

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finite no. of elements $< \lambda - \epsilon$.

<u>3</u> ~~is~~ $\overline{\lim} S = \mu$, $\underline{\lim} S = \lambda$.

<u>4</u> If $m = \text{glb } S$, $M = \text{lub } S$, then all the elements of S lie in $[m, M]$, not necessarily in $[\lambda, \mu]$. $m \le \lambda \le \mu \le M$.

<u>Ex</u> Let $S = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \cdots \frac{1}{n} \cdots\}$.

$S' = \{0\}$. $S' \subset S$. S is closed.

$T = \{1, \frac{1}{2}, \frac{1}{3} \cdots \frac{1}{n} \cdots\}$

$T' = \{0\}$ T' ~~ϕ~~ is not a subset of T

$\therefore$ T is not closed set.

<u>Ex</u> Every pt of the closed interval $[a, b]$ is a cluster pt, and each such pt $\in [a, b]$. & hence $[a, b]$ is a closed set.

$[a, b]$ is a perfect set.

An open interval $]a, b[$, however, is a closed set. Besides each interior pt of (a, b) the two end pts a and b (though they do not belong to the set) are limit pts. Thus the derived set of (a, b) is $[a, b]$ which is not a subset of (a, b).

<u>Ex</u> $S = \{1 + \frac{1}{n} : n \in N\} \cup \{-1 - \frac{1}{n} : n \in N\}$

$S = \{2, -2, 3/2, -3/2, \frac{4}{3}, -\frac{4}{3} \cdots\}$

The set is bounded, $m = -2$, $M = 2$.

$\lambda = -1$, $\mu = 1$

$S' = \{-1, 1\}$ $\therefore$ S is not closed set.

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4. $S = \{-1 - \frac{1}{n} : n \in N\} \cup \{-\frac{1}{n} : n \in N\}$

i.e. $S = \{-2, -1, -3/2, -\frac{1}{2}, -\frac{4}{3}, -1/3 \cdots\}$

In this bounded set $m = -2$, $M = 0$

$\lambda = -1$, $\mu = 0$.

$m < \lambda < \mu = M$

$S' = \{-1, 0\}$ S is not closed.

5. $S = \{\frac{1}{n} : n \in N\} \cup \{1 + \frac{1}{n} : n \in N\}$

$S = \{1, 2, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{4}{3} \cdots\}$

$m = 0$, $M = 2$, $\lambda = 0$, $\mu = 1$

$m = \lambda < \mu < M$

$S' = \{0, 1\}$ S is not closed set.

6. $S = \{-2, 2\} \cup \{\frac{1}{n} : n \in N\} \cup \{-1/n : n \in N\}$

$S = \{-2, 2, -1, 1, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, \cdots\}$

Here $m = -2$, $M = 2$; $\lambda = \mu = 0$. Thus

$m < \lambda = \mu < M$.

$S' = \{0\}$ and S is not a closed set.

7. $S = \{-1 + \frac{1}{n} ; n \in N\} \cup \{1 - \frac{1}{n} : n \in N\}$

$S = \{0, -\frac{1}{2}, \frac{1}{2}, -\frac{2}{3}, \frac{2}{3}, -\frac{3}{4}, \frac{3}{4} \cdots\}$

$m = -1$, $M = 1$, $\lambda = -1$, $\mu = 1$ Thus

$m = -\lambda < \mu = M$.

$S' = \{-1, 1\}$ and S is not a closed set

8. Construct an example of a set no element of which lies between its upper and lower limits.

The set $= \{-\frac{1}{n} : n \in N\} \cup \{1 + \frac{1}{n} ; n \in N\}$

$= \{-1, 2, -\frac{1}{2}, \frac{3}{2}, -\frac{1}{3}, \frac{4}{3} \cdots\}$

Here $\lambda = 0$, $\mu = 1$, The elements crowd ~~between~~ near 0, & 1 but no element lies between 0 & 1.

Ex Find a set with only $\sqrt{2}$ is as its limit pt,

Ans. $\left\{ \sqrt{2} + \frac{1}{n} , n \in N \right\}$

Ex Find the derived sets of the following sets :-

i) $\left\{ 1 + \frac{(-1)^n}{n} : n \in N \right\}$; ii) $\left\{ \frac{1+(-1)^n}{n} : n \in N \right\}$

iii) $\left\{ (-1)^n + \frac{1}{n} : n \in N \right\}$, iv) $\left\{ 2^n + \frac{1}{2^n} : n \in N \right\}$