SAMUELSON MULTIPLIER-ACCELERATION INTERACTION MODEL

A nice application of second order linear difference equation has been made by Samuelson in his famous multiplier-acceleration model. This model describes how, as a result of simultaneous functioning of the multiplier and the accelerator, a continuous process of expansion of income is generated. Suppose a change in autonomous investment takes place in an economy in a certain period. It will then induce the multiplier to work, which results in a change in income. A change in income, in turn, will induce investment and the accelerator will start operating. This again induces investment and the entire process will start a new round. The model can be described by the following system of equations.

$$Y_t = C_t + I_t + G_0$$
$$C_t = \gamma Y_{t-1} \qquad (0<\gamma<1)$$
$$I_t = \alpha (C_t - C_{t-1}) \qquad (\alpha>0)$$

where γ = MPC

α = Accelerator or accelerator Coefficient

$$I_t = \alpha(\gamma Y_{t-1} - \gamma y_{t-2}) = \alpha\gamma(y_{t-1} - y_{t-2})$$

$$\therefore Y_t = \gamma Y_{t-1} + \alpha\gamma(y_{t-1} - y_{t-2}) + G_0$$

or, $y_t = (1+\alpha)\gamma y_{t-1} - \alpha\gamma y_{t-2} + G_0$

or, $y_t - (1+\alpha)\gamma y_{t-1} + \alpha\gamma y_{t-2} = G_0$

After shifting the subscripts forward by two periods.

$$y_{t+2} - (1+\alpha)\gamma y_{t-1} + \alpha\gamma y_{t-2} = G_0$$

Particular solution

$$y_t = K$$
$$y_{t+1} = K$$
$$y_{t+2} = K$$

$$\therefore K - (1+\alpha)\gamma K + \alpha\gamma K = G_0$$
$$\therefore K(1-\gamma-\alpha\gamma+\alpha\gamma) = G_0$$
$$\therefore K = \frac{G_0}{1-\gamma}$$

Intertemporal equilibrium income $(\bar{y})$

= Particular Solution (y_p)

$$= \frac{G_0}{1-\gamma}$$

Complementary Function

Reduced form $y_{t+2} - \gamma(1+\alpha) y_{t+1} + \alpha\gamma y_t = 0$

$$y_t = Ab^t$$
$$y_{t+1} = Ab^{t+1}$$
$$y_{t+2} = Ab^{t+2}$$

$$Ab^{t+2} - \gamma(1+\alpha) Ab^{t+1} + \alpha\gamma Ab^t = 0$$

$$\therefore Ab^t \left(b^2 - \gamma(1+\alpha)b + \alpha\gamma\right) = 0$$

$$\therefore b^2 - \gamma(1+\alpha)b + \alpha\gamma = 0 \quad \text{as } Ab^t \neq 0$$

$$b_1, b_2 = \frac{+\gamma(1+\alpha) \pm \sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}}{2}$$

There are three possible Cases

Case-1:- Real and distinct roots

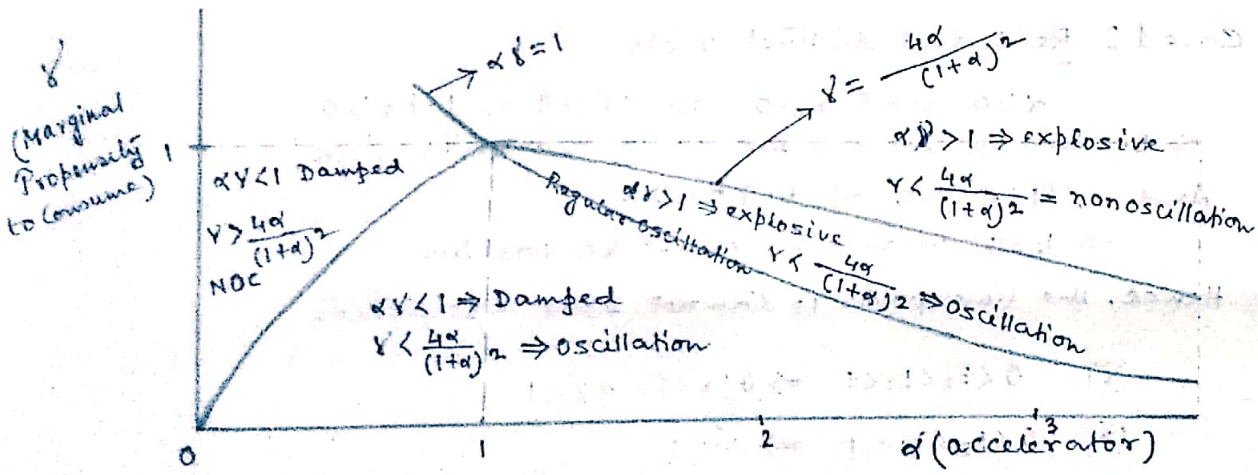
$$\gamma^2(1+\alpha)^2 > 4\alpha\gamma$$
$$\Rightarrow \gamma > \frac{4\alpha}{(1+\alpha)^2}$$

Case-2: Repeated root

$$\gamma = \frac{4\alpha}{(1+\alpha)^2}$$

Case-3: Complex roots $\Rightarrow \gamma < \frac{4\alpha}{(1+\alpha)^2}$

In the following figure we have drawn the graph of the equation $\gamma = 4\alpha/(1+\alpha)^2$. It is clear that the (α, γ) pairs that are located exactly on this curve pertain to Case-2. On the other hand, (α, γ) pairs lying above this curve (involving higher γ values) have to do with Case-1, and those lying below the curve with Case-3.



The above figure reveals clearly the conditions under which cyclical fluctuations can emerge endogenously from the interaction of the multiplier and accelerator. But this tells nothing about the convergence or divergence of the time path of y.

Convergence vs. Divergence

$$b^2 - \gamma(1+\alpha)b + \alpha\gamma = 0$$

$$b_1, b_2 = \frac{\gamma(1+\alpha) \pm \sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}}{2}$$

Since the question of convergence versus divergence depends on the values of b_1 and b_2.

b_1 and b_2 depend on the values of the parameter α and γ.
$\Rightarrow$ the conditions of convergence and divergence should be expressible in terms of the values of α and γ. Now we have to find the b_1 and b_2 with the parameters α and γ.

$$b_1 + b_2 = \frac{\gamma(1+\alpha)}{2} + \frac{\sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}}{2} + \frac{\gamma(1+\alpha)}{2} - \frac{\sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}}{2}$$

$$= \gamma(1+\alpha)$$

$$b_1 b_2 = \left(\frac{\gamma(1+\alpha)}{2} + \frac{\sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}}{2}\right)\left(\frac{\gamma(1+\alpha)}{2} - \frac{\sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}}{2}\right)$$

$$= \frac{\gamma^2(1+\alpha)^2}{4} - \frac{\gamma^2(1+\alpha)^2}{4} + \frac{4\alpha\gamma}{4}$$

$$= \alpha\gamma$$

On the basis of these two equations, we may observe that

$$(1-b_1)(1-b_2) = 1 - (b_1 + b_2) + b_1 b_2$$

$$= 1 - \gamma(1+\alpha) + \alpha\gamma$$

$$= 1 - \gamma - \alpha\gamma + \alpha\gamma$$

$$= 1 - \gamma$$

Since $0 < \gamma < 1$

$$0 < (1-b_1)(1-b_2) < 1 \quad \cdots\cdots\cdots\cdots \text{(A)}$$

Case 1: Real and distinct roots

$\alpha > 0$ and $\gamma > 0 \Rightarrow \alpha\gamma > 0 \Rightarrow b_1 b_2 > 0$

$\Rightarrow$ b_1 and b_2 possess the same algebraic sign.

And $\gamma(1+\alpha) > 0 \Rightarrow b_1 + b_2 > 0$

$\Rightarrow$ Both b_1 and b_2 must be positive.

Hence, the time path Y_t cannot have oscillation.

(i) $0 < b_2 < b_1 < 1 \Rightarrow 0 < \gamma < 1, \ \alpha\gamma < 1$

(ii) $0 < b_2 < b_1 = 1 \Rightarrow \gamma = 1$

(iii) $0 < b_2 < 1 < b_1 \Rightarrow \gamma > 1$

(iv) $1 = b_2 < b_1 \Rightarrow \gamma = 1$

(v) $1 < b_2 < b_1 \Rightarrow 0 < \gamma < 1, \ \alpha\gamma > 1$

(i) Both b_1 and b_2 are positive fraction and satisfies (A)

$\Rightarrow$ Conforms that $0 < \gamma < 1$

Since $b_1 b_2 < 1$

$\Rightarrow \alpha\gamma < 1 \Rightarrow$ Convergent

Possibilities (ii), (iii) and (iv) violate the Condition (A) and result in inadmissible γ values.

Hence they must be ruled out.

But in Possibility (v), both b_1 and b_2 greater than one, may still be satisfied if $(1-b_1)(1-b_2) < 1$

Here we have $\alpha\gamma > 1 \Rightarrow$ Divergent

Case 2: Repeated roots

Roots are $b = \frac{\gamma(1+\alpha)}{2} > 0$ as $\alpha > 0, \gamma > 0$

Thus, there is again no oscillation.

(vi) $0 < b < 1 \Rightarrow \gamma < 1$; $\alpha\gamma < 1$

(vii) $b = 1 \Rightarrow \gamma = 1$

(viii) $b > 1 \Rightarrow \gamma < 1$; $\alpha\gamma > 1$

Under possibility (vi), b is a positive fraction

$\alpha\gamma < 1 \Rightarrow$ Convergent

Possibility (viii) $\Rightarrow b > 1 \Rightarrow \gamma\alpha > 1 \Rightarrow$ Divergent

Possibility (vii) violates the Condition (A)

Case 3: Complex roots

We have stepped fluctuation and hence endogeneous business cycles.

Here. $h = \frac{\gamma(1+\alpha)}{2}$ $\quad u = \frac{\sqrt{4\alpha\gamma - \gamma^2(1+\alpha)^2}}{2}$

$$r^2 = h^2 + u^2 = \frac{\gamma^2(1+\alpha)^2}{4} + \frac{4\alpha\gamma}{4} - \frac{\gamma^2(1+\alpha)^2}{4}$$

$$r^2 = \alpha\gamma$$

$$\therefore r = \sqrt{\alpha\gamma}$$

(ix) $r < 1 \Rightarrow \alpha\gamma < 1 \Rightarrow$ Convergent

(x) $r = 1 \Rightarrow \alpha\gamma = 1 \Rightarrow$ Regular oscillation.

(xi) $r > 1 \Rightarrow \alpha\gamma > 1 \Rightarrow$ Divergent