Exogenous (old) Growth Theory: Phase II

The Solow Model: Neo-classical Theory of Growth

Aggregate output in period t is given by the production function

$$Y_t = F(K_t, L_t)$$

we rewrite ~~...~~ it without subscript 't' to avoid ~~...~~ clumsiness of notations

$$Y = F(K, L) \quad \text{——— (1)}$$

K = capital
L = Labour

$$F(0, L) = 0, \; F(K, 0) = 0$$

$$F_K > 0, \; F_L > 0, \; F_{KK} < 0, \; F_{LL} < 0$$

$$\left.\begin{aligned} \lim_{K \to 0} F_K(K, L) &= \lim_{L \to 0} F_L(K, L) = \infty \\ \lim_{K \to \infty} F_K(K, L) &= \lim_{L \to \infty} F_L(K, L) = 0 \end{aligned}\right\} \text{Inada conditions.}$$

~~...~~ It is assumed that the production function exhibits CRS i.e.

$$\lambda Y = F(\lambda K, \lambda L) \quad \forall \lambda > 0$$

Choosing $\lambda = \frac{1}{L} > 0$ we have

$$\frac{Y}{L} = F\left(\frac{K}{L}, 1\right)$$

$$\Rightarrow \quad y = f(k) \quad \text{——— (2)}$$

where $y = \frac{Y}{L}$, $k = \frac{K}{L}$

and $f(\cdot) \Rightarrow$ intensive production function.

i.e. the neo-classical production function is expressed in percapita terms.

$$f' = F_K > 0$$

$$f'' = L F_{KK} < 0$$

$$\left.\begin{aligned} &\lim_{k \to 0} f' = \infty \\ \text{and} \quad &\lim_{k \to \infty} f' = 0 \end{aligned}\right\} \text{ Inada conditions}$$

$$y = f(k)$$

$$\Rightarrow \frac{Y}{L} = f(k)$$

$$\text{or, } Y = L f(k)$$

$$\frac{\partial Y}{\partial K} = L f'(k) \frac{\partial k}{\partial K} = L f'(k) . \frac{1}{L}$$

$$\Rightarrow \frac{\partial Y}{\partial K} = f'(k) \quad \text{———} (3)$$

$$\frac{\partial Y}{\partial L} = \frac{\partial [L f(k)]}{\partial L} = f(k) \frac{\partial L}{\partial L} + L \frac{\partial f(k)}{\partial k} \frac{\partial k}{\partial L}$$

$$= f(k) + L f'(k) \left(-\frac{K}{L^2}\right)$$

$$\Rightarrow \frac{\partial Y}{\partial L} = f(k) - k f'(k) \quad \text{———} (4)$$

We now consider the fundamental equation of Solow model. For this purpose we assume that a fraction of income is saved and is invested to augment capital stock. Thus

$$\dot{K} = sY - \delta K \quad \text{———} (5)$$

where $\dot{K} = \frac{dK}{dt}$, δ = rate of depreciation of capital

s = saving propensity, Y = output or national [illegible]

As $k = \frac{K}{L}$

$$\ln k = \ln \frac{K}{L} = \ln K - \ln L$$

$$\Rightarrow \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$$

We consider $\frac{\dot{L}}{L} = n$ → the natural rate of growth (exogenously given)

$$\dot{K} = sY - \delta K$$

(where $s \Rightarrow$ marginal propensity to save
$\delta \Rightarrow$ rate of depreciation of capital stock)

$$\Rightarrow \frac{\dot{K}}{K} = s\frac{Y}{K} - \delta$$

$$\therefore \frac{\dot{k}}{k} = s\frac{Y}{K} - \delta - n$$

$$\text{or, } \dot{k} = sk.\frac{Y}{K} - \delta k - nk$$

$$\text{or, } \dot{k} = s.\frac{Y}{L} - (\delta + n)k$$

$$\text{or, } \dot{k} = s\,f(k) - (\delta + n)k \quad \text{———} (6)$$

Equation (6) is called the fundamental differential equation of the Solow model.

We can explain equation (6) in terms of figure 1. In figure 1 the intensive production function is concave to the horizontal axis as $f'' < 0$. The term $sf(k)$ in equation (6) has the shape of a production function but it lies below the production function as $0 < s < 1$. Inada conditions imply $sf(k)$ curve is vertical at $k = 0$ and becomes flat when $k \to \infty$

The $(n+\delta)k$ ~~asees~~ term as appears in equation (6) is drawn as a straight line with positive slope $(n+\delta)$ in figure 1.

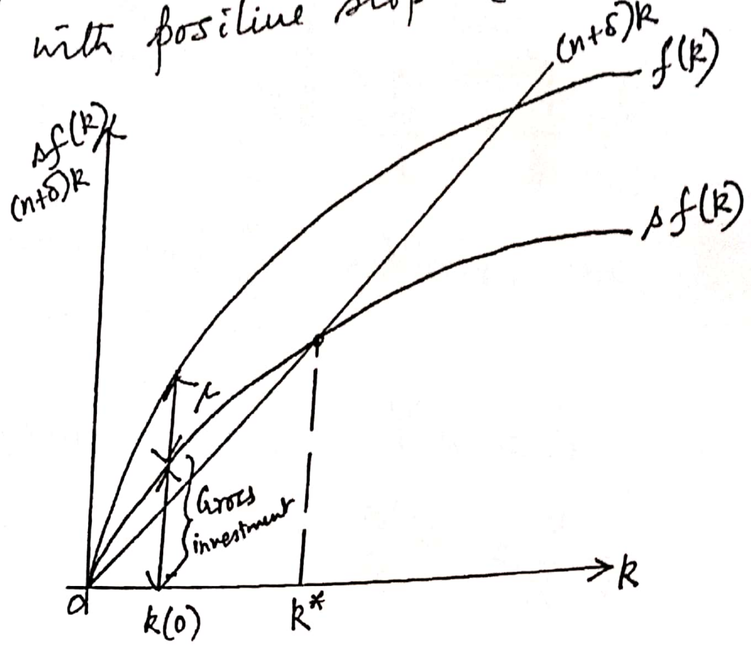

Figure 1

In figure 1 gross investment per person at initial capital stock per person $k(0)$ is $sf(k)$. Hence consumption per person at this point is given by the vertical difference between $f(k)$ and $sf(k)$. Thus $c = (1-s)f(k)$.

[A digression: Microfoundation of the fundamental equation of Solow model

we first consider the household side. Households own financial assets and labour. The rate of return of assets (r: rate of interest) is r and the rate of return of labour is the wage rate w. Households use the income that they do not consume to accumulate assets.

Thus,
$$\frac{d(\text{assets})}{dt} = [r.(\text{assets}) + w.L] - C \qquad (7)$$

Let assets per person be

$$a = \frac{\text{assets own by a household}}{L}$$

Let $c = \frac{C}{L}$ = consumption per person

$$\frac{d(\text{assets}/L)}{dt} = \frac{1}{L}\left[r(\text{assets}) + w.L - C\right] - \frac{\text{assets}}{L} \cdot \frac{1}{L}\frac{dL}{dt}$$

or, $\frac{da}{dt} = (ra + w) - c - na$

or, $\dot{a} = (ra + w) - c - na$ —— (8)

If we focus on the firm side we have R as the rental price for a unit of capital services. We consider rate of depreciation of capital as δ. The net rate of returns to a household that owns a unit of capital is $R - \delta$. Households also receive the interest rate r on funds lent to other households. Capital and loans are considered as perfect substitutes. Thus $r = R - \delta$ or $R = r + \delta$

Profit of the representative firm is

$$\pi = F(K, L) - (r+\delta)K - wL \quad \text{—— (9)}$$

[price of the product is normalized to unity]

CRS $\Rightarrow$

$$\pi = L\left[f(k) - (r+\delta)k - w\right] \quad \text{—— (10)}$$

Profit maximization w.r.t. k, for given L, implies

$$f'(k) = r + \delta \qquad (11)$$

i.e. MP_K = rental price

Under CRS we find product exhaustion theorem is valid which implies profit (or longrun profit) is zero.

For profit to be zero the wage rate has to be equal to the marginal product of labour. Using equation (11) we find for $\pi = 0$ in (10) we should have

$$f(k) - kf'(k) = w \qquad (12)$$

The LHS of equation (12) implies MP_L

In a closed economy at equilibrium as all borrowing and lending cancel in an economy the only asset is capital. Hence equilibrium in the asset market implies $a = k$.

Thus substituting $a = k$, $r = f'(k) - \delta$ and $w = f(k) - kf'(k)$ in equation (8) we get

$$\dot{k} = [(f'(k) - \delta)k + f(k) - kf'(k)] - c - nk$$

$$\text{or, } \dot{k} = f(k) - \delta k - c - nk$$

$$\text{or, } \dot{k} = f(k) - c - (n+\delta)k$$

As $c = (1-s)f(k)$ we have

$$\dot{k} = sf(k) - (n+\delta)k$$

It is nothing but fundamental equation (equation (6)) of Solow model].

The steady state in Solow Model

Steady state means the rate of growth of capital, the rate of growth of output, the rate of growth of consumption all grow at the same rate and it is equal to the rate of growth of labour force. Thus steady-state implies

$$\frac{\dot{K}}{K} = \frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{\dot{L}}{L}$$

Here $\frac{\dot{L}}{L} = n$ (exogenously given)

Thus $$\frac{\dot{K}}{K} = \frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{\dot{L}}{L} = n$$

It implies $\frac{\dot{k}}{k} = 0$ (as $k = \frac{K}{L}$) at steady-state

$\frac{\dot{k}}{k} = 0 \Rightarrow \dot{k} = 0$ and when $\dot{k} = 0$ we have

$$sf(k) = (n+\delta)k \quad \text{——— (13)}$$

From equation (13) we can uniquely solve for k and we denote it as k^*. k^* is the steady-state value of k which satisfies equation (13)

$$sf(k^*) = (n+\delta)k^* \quad \text{——— (13.1)}$$

Since k is a constant in the steady state, y and c are also constant at the values $y^* = f(k^*)$ and $c^* = (1-s)f(k^*)$, respectively. Hence, in the neo-classical model, the per-capita quantities k, y and c do not grow in the steady-state.

In figure 1 the steady-state level of k (given by k^*) is shown on the basis of intersection of $sf(k)$ curve and $(n+\delta)k$ line.

When s increases the $sf(k)$ curve shifts upward and the level of ~~k increases~~ steady-state k increases. When n or δ increases the line $(n+\delta)k$ becomes more steeper so that k^* falls.

The Golden Rule of Capital Accumulation and Dynamic Inefficiency in Solow Model

The steady-state value of k, i.e. k^*, is achieved for given value of s. As s increases the value of k^* increases. Thus, $k^* = k^*(s)$ with $\frac{dk^*(s)}{ds} > 0$.

The steady-state level of per capita consumption is $c^* = (1-s)f[k^*(s)]$

At steady-state we have

$$sf(k^*) = (n+\delta)k^*$$

$$\Rightarrow \quad c^*(s) = f[k^*(s)] - (n+\delta)k^*(s) \quad \text{—— (14)}$$

(14) $\Rightarrow$

$$\frac{dc^*}{ds} = f'(k^*)\frac{dk^*}{ds} - (n+\delta)\frac{dk^*}{ds}$$

$$\text{or,} \quad \frac{dc^*}{ds} = [f'(k^*) - (n+\delta)]\frac{dk^*}{ds} \quad \text{—— (15)}$$

Since $\frac{dk^*}{ds} > 0$ $\quad \frac{dc^*}{ds} \gtreqless 0$ according as $f'(k^*) \gtreqless (n+\delta)$

Initially $f'(k^*)$ is high but as k^* increases $f'(k^*)$ falls due to diminishing marginal productivity i.e. $f''(k^*) < 0$.

Therefore we find that as s increases (implying k^* increases) c^* increases, then reaches a maximum and after that it falls. This is shown in figures 2(a) and 2(b)

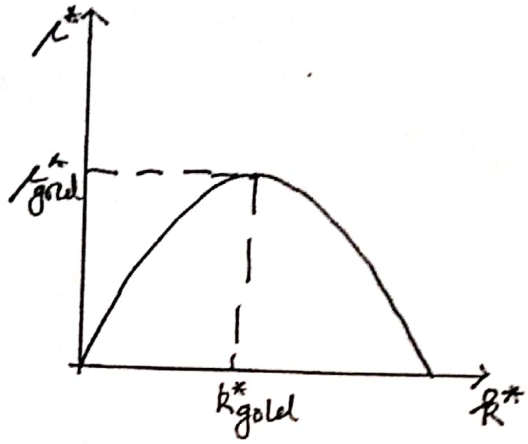

Fig 2(a)

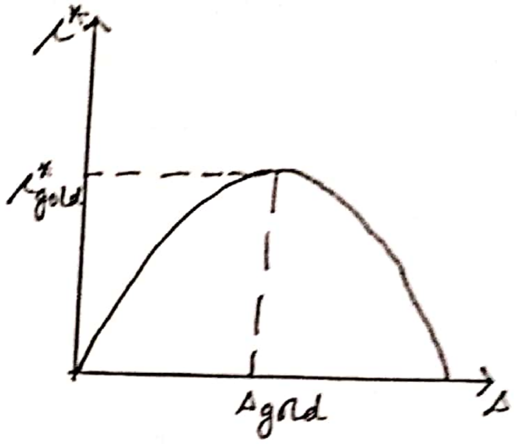

Fig 2(b)

The maximum per capita consumption at steady state is achieved when $k^* = k^*_{gold}$. The condition corresponding to this is given by

$$f'(k^*_{gold}) = n + \delta \qquad \text{——— (16)}$$

$$\Rightarrow f'(k^*_{gold}) - \delta = n \qquad \text{——— (17)}$$

Thus, the condition for maximum steady-state per capita consumption is achieved when the marginal productivity of capital net of rate of depreciation of capital (or net marginal productivity of capital) is equal to the rate of growth of population.

The condition in equation (17) is called the 'golden rule of capital accumulation' (as shown by Phelps). It implies that if we provide the same amount of consumption to members of each current and future generation — that is, if we do not provide less to future generations than to ourselves — then maximum amount of per capita consumption is c^*_{gold} (60)

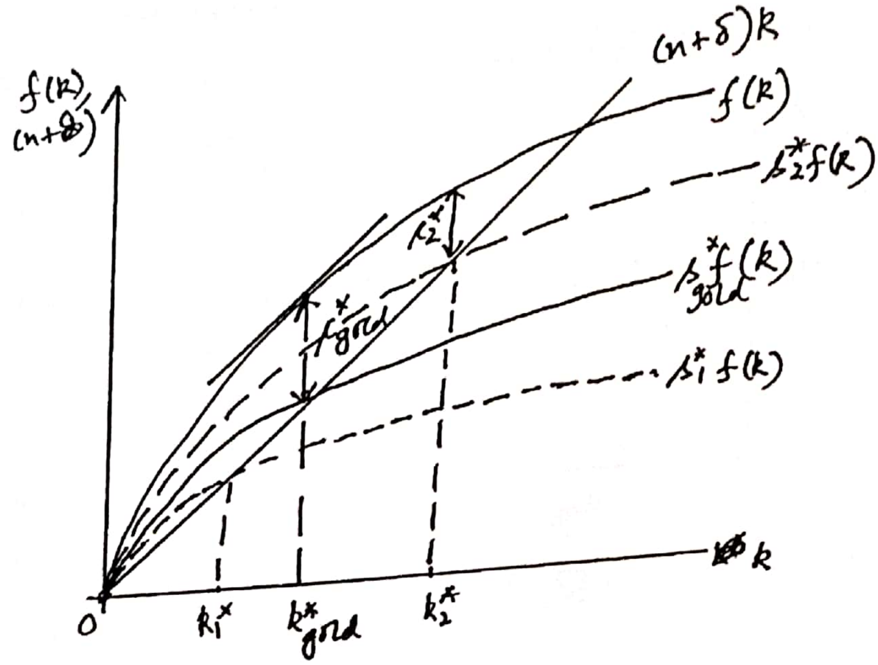

Fig 3

Figure 3 shows that k^*_{gold} is the golden level of capital. It is the steady-state value ~~of~~ of ~~maximizes~~ k that maximises consumption. When $k = k^*_{gold}$ not only $sf(k^*_{gold}) = (n+\delta)k^*_{gold}$ but also $f'(k^*_{gold}) = (n+\delta)$. At this point the tangent to the production function parallels the $(n+\delta)k$ line. From the figure we find that as s^* increases k^* increases and when s falls k^* falls. Thus from figure 3 we find that $s_1^* < s^*_{gold} < s_2^*$ implies $k_1^* < k^*_{gold} < k_2^*$

We consider an economy, such as the one described by the saving rate s_2^* in figure 3 for which $s_2^* > s^*_{gold}$, so that $k_2^* > k^*_{gold}$ and $c_2^* < c^*_{gold}$. Suppose we start from a situation when $s^* = s_2^*$ and $c_2 = c_2^*$. Then the level of c

falls monotonically during the transition towards its new steady-state value c^*_{gold}.
Since $c_2^* < c^*_{gold}$ we conclude that c exceeds its previous value, c_2^*, at all transitional dates, as well as in the new steady-state.
Hence, when $s > s^*_{gold}$, the economy is oversaving in the sense that per capita consumption at all points of time could be raised by lowering the saving rate. An economy that oversaves is said to be dynamically inefficient, because the path of per capita consumption lies below feasible alternative paths at all points of time.

Transitional Dynamics

The transitional dynamics shows how an economy's per capita income converges towards its own steady-state value (and to the per capita incomes of other economies).

Division of both sides of equation (6) by k we get

$$\gamma_K = \frac{\dot{k}}{k} = \frac{s f(k)}{k} - (n+\delta) \qquad \text{———} \quad (18)$$

$\gamma_K \rightarrow$ rate of growth of k

In equation (18) as $\dfrac{d \frac{f(k)}{k}}{dk} = \dfrac{k f'(k) - f(k)}{k^2}$

$$= -\frac{[f(k) - k f'(k)]}{k^2}$$

$$= -\frac{MP_L}{k^2} < 0$$

In Solow model as $AP_K = \frac{f(k)}{k}$

$= AP_K$

falls as k increases

$\frac{d f(k)}{dk} < 0$.

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Thus $s\frac{f(k)}{k}$ falls as k increases.

We can refer to $s\frac{f(k)}{k}$ as the savings curve and $(n+\delta)$ as the depreciation curve. This is shown in figure 4.

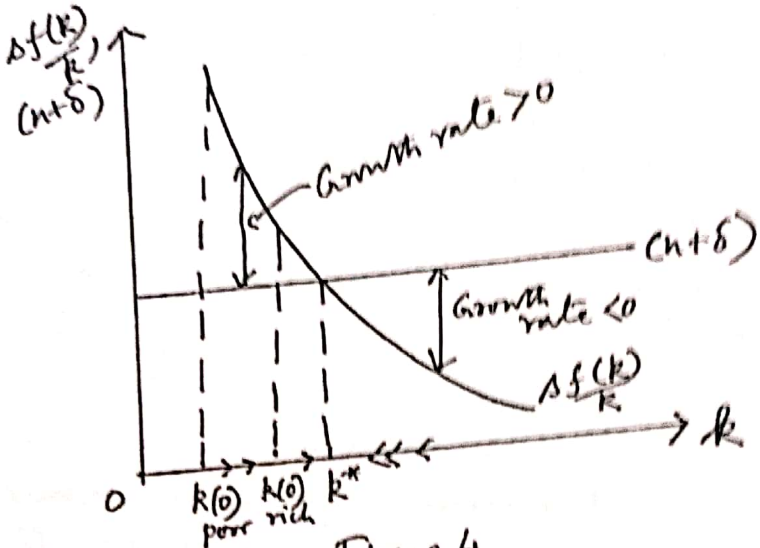

Figure 4

In figure 4 the steady-state level of k is k^*. To the left of k^*, $s\frac{f(k)}{k}$ curve lies above $(n+\delta)$ so that the growth rate is positive. Hence k rises over time. As k increases, $\frac{\dot{k}}{k}$ declines and approaches 0 as k approaches k^*. (The saving curve gets closer to the depreciation line as k gets closer to k^*; hence $\frac{\dot{k}}{k}$ falls). The reason behind this declining growth rate along the transition is the existence of diminishing returns to capital. An analogous argument demonstrates that if the economy starts above the steady-state, $k(0) > k^*$, then the growth rate of k is negative and k falls over time. Thus the system is globally stable.

We can also study the behaviour of output along the transitional dynamics.

The growth rate of output per capita is derived as follows.

$$y = f(k)$$

$$\Rightarrow \ln y = \ln f(k)$$

$$\Rightarrow \frac{\dot{y}}{y} = \frac{\dot{f}(k)}{f(k)}$$

or, $$\frac{\dot{y}}{y} = \frac{df(k)}{dk} \cdot \frac{dk}{dt} \cdot \frac{1}{f(k)}$$

or, $$\frac{\dot{y}}{y} = f'(k) \cdot \frac{\dot{k}}{f(k)}$$

or, $$\frac{\dot{y}}{y} = \left[f'(k) \cdot \frac{k}{f(k)}\right]\frac{\dot{k}}{k} \quad \text{——— (19)}$$

In equation (19) $f'(k) \cdot \frac{k}{k}$ is the capital share or the share of rental income on capital in total income (it is also the elasticity of $f(k)$ with respect to k or the output elasticity with respect to capital)

In case of Cobb-Douglas production function we have $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$ so that the capital share is a constant α. In this case

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} \quad \text{——— (19.1)}$$

Thus the behaviour of $\frac{\dot{y}}{y}$ is same as that of the behaviour of $\frac{\dot{k}}{k}$

Absolute and Conditional Convergence

From the fundamental equation of Solow model we have derived equation (18) which is stated as

$$\gamma_k = \frac{\dot{k}}{k} = \frac{sf(k)}{k} - (n+\delta)$$

$\gamma_k \Rightarrow$ rate of growth of k

$$\frac{\partial[\dot{k}/k]}{\partial k} = \frac{s\,\partial f(k)/k}{\partial k}$$

$$= s\,\frac{kf'(k) - f(k)}{k^2}$$

$$= \frac{s\left[f'(k) - \frac{f(k)}{k}\right]}{k} < 0 \qquad \text{—— (20)}$$

[Note $f(k) - kf'(k) = MP_L$

$\Rightarrow \dfrac{kf'(k) - f(k)}{k^2} = -\dfrac{MP_L}{k^2} < 0$]

(20) $\Rightarrow$ smaller values of k are associated with larger values of $\frac{\dot{k}}{k}$

An important question that arises is: Does this result mean that economies with lower capital per person tend to grow faster in per capita terms? In other words, does there tend to be <u>convergence</u> across economies?

To answer these questions we consider a group of closed economies (say isolated regions or countries) that are structurally similar in the sense that they have the same values of the parameters s, n and δ and also have the same production function $f(.)$. Thus the economies have the same steady-state values k^* and y^*. The only difference among the countries is the initial quantity of capital per person $k(0)$. In figure 4 we have distinguished between two types of economies: one with lower initial value of $k(0)$, denoted as $k(0)_{poor}$, and the other with high initial value of $k(0)$, denoted as $k(0)_{rich}$. Since each economy has the same underlying parameters, the dynamics of k are determined in each case by the same $\frac{sf(k)}{k}$ and $(n+\delta)$ curves.

Hence, the growth rate $\frac{\dot{k}}{k}$ is unambiguously higher for the economy with lower initial $k(0)$ value, i.e. $k(0)_{poor}$. Thus regions or countries with lower starting values of capital-labour ratio also have higher per capita growth rates $\frac{\dot{k}}{k}$ and thereby tend to catch up or converge to those with higher capital-labour ratios.

The hypothesis that poor economies tend to grow faster per capita than the rich ones - without conditioning on any other characteristics of economies - is referred to as absolute convergence or absolute β- convergence. Empirically it has mixed reviews when confronted with data on group of economies.

If we modify the analysis by dropping the assumption that all economies have the same parameters and therefore have the same steady-state positions, we can develop the concept of conditional convergence. The main idea is that the economy grows faster the further it is from its own steady-state value. This is illustrated in figure 5.

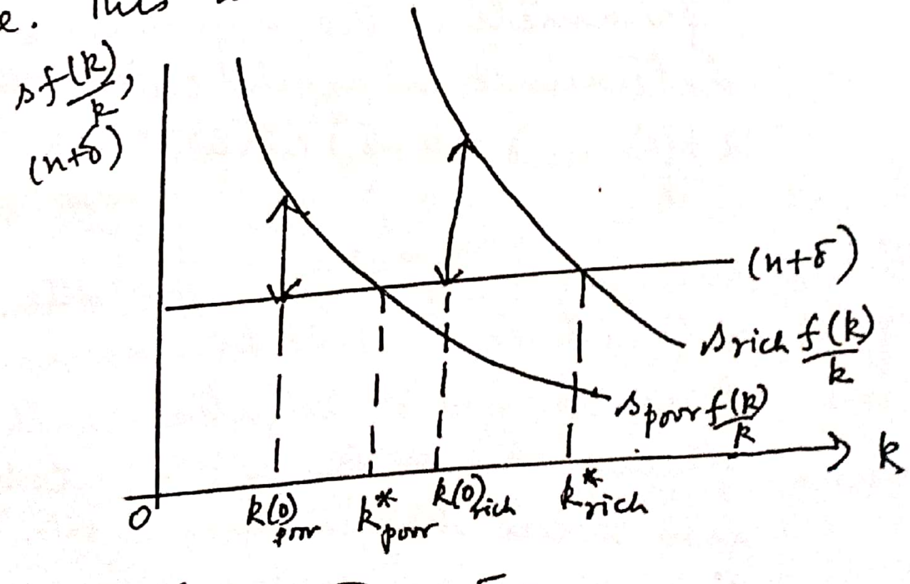

Figure 5

In figure 5, we consider two economies that differ in two respects:

(i) $k(0)_{poor} < k(0)_{rich}$

(ii) $A_{poor} \neq A_{rich}$ and $A_{poor} < A_{rich}$

From the figure we thus find

$$k^*_{poor} \neq k^*_{rich}$$

In fact $k^*_{poor} < k^*_{rich}$

From figure 5 we find at the initial levels of k

$$\left(\frac{\dot{k}}{k}\right)_{poor} < \left(\frac{\dot{k}}{k}\right)_{rich}$$

$\Rightarrow$ If the rich country has a higher initial capital-labour ratio and also has a higher saving rate then it grows faster than the poor country. However, each economy converges to its own steady-state ~~value~~

This is the concept of conditional convergence or conditional β-convergence.

When k is low $\frac{f(k)}{k}$ or AP_K is high. So for the poor country initially $\frac{f(k)}{k}$ is high. However, the low saving rate of the poor country offsets its higher average productivity of capital as a determinant of economic growth. Hence the poor economy may grow at a slower rate than the rich one.

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Algebraically we can illustrate the concept of conditional convergence
From equation (13·1) we ~~get~~

$$s = (n+\delta)\frac{k^*}{f(k^*)}$$

Hence equation (18) implies

$$\frac{\dot{k}}{k} = (n+\delta)\left[\frac{s\,f(k)/k}{(n+\delta)} - 1\right]$$

$$\text{or, } \frac{\dot{k}}{k} = (n+\delta)\left[\frac{f(k)/k}{f(k^*)/k^*} - 1\right] \quad \text{——— (21)}$$

when $k = k^*$ we have $\frac{\dot{k}}{k} = 0$

For given k^*, a ~~lower k^* implies hig~~ a lower k implies a higher $\frac{f(k)}{k}$
The lower is k compared to k^*, the more will be the level of $\frac{f(k)}{k}$ and higher will be $\frac{\dot{k}}{k}$. So if the initial value of k relatively has a shorter distance from k^* in case of poor countries then $f(k)/k$ will be relatively less compared to $f(k^*)/k^*$ and $\frac{\dot{k}}{k}$ will be relatively less

For a Cobb Douglas production function

$$Y = AK^{\alpha}L^{1-\alpha}$$

$$y = Ak^{\alpha}$$

$$\frac{y}{k} = Ak^{(\alpha-1)} \Rightarrow s = \frac{(n+\delta)}{A}k^{*(1-\alpha)}$$

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Hence equation (21) implies

$$\frac{\dot{k}}{k} = (n+\delta)\left[\frac{s\,A k^{(\alpha-1)}}{(n+\delta)} - 1\right]$$

$$\text{or, } \frac{\dot{k}}{k} = (n+\delta)\left[\left\{\left(\frac{1}{A}\right)k^{*(1-\alpha)}\right\}A k^{\alpha-1} - 1\right]$$

$$\left[\because \frac{s}{(n+\delta)} = \left\{\left(\frac{1}{A}\right)k^{*(1-\alpha)}\right\}\right]$$

$$\text{or, } \frac{\dot{k}}{k} = (n+\delta)\left[\left(\frac{k}{k^*}\right)^{\alpha-1} - 1\right] \qquad (22)$$

Hence, the growth rate of capital, k, depends on the ratio $\frac{k}{k^*}$, that is, it depends on the distance between the current and the steady-state capital-labour ratio.

Technological Progress in the Solow Model

The section deals with exogenous ~~technical~~ technological progress in Solow model. The progress may allow producers to generate the same amount of output with relatively either with less capital input (capital-saving) or with less labour input (labour saving) technological progress. Inventions that do not save relatively more of either input are called neutral or unbiased.

There are three popular definitions of technical progress and they are due to Hicks, Harrod and Solow.

A technological progress is Hicks neutral if the ratio of marginal products remain unchanged for a given capital-labour ratio.

Let $T(t)$ be the level of technology at time t, $\dot{T}(t) \geqq 0$. Incorporating technological progress we can rewrite the production function as

$$Y = F(K, L, T)$$

Hicks-neutral technological progress implies

$$Y = T(t)\, F(K, L) \qquad (23)$$

Harrod defines a technological progress as neutral (Harrod neutral) if the relative input shares, $\frac{KF_K}{LF_L}$, remain unchanged for a given capital-output ratio. Robinson and Uzawa have shown that this definition follows from the production function

$$Y = F[K, L.T(t)] \qquad (24)$$

Thus Harrod-neutral technical or technological progress implies it is labour-augmenting.

Solow defines a technological progress as neutral (Solow neutral) if the relative input shares, $\frac{LF_L}{KF_K}$, remain unchanged for a given labour-output ratio.

In case of Solow neutral technical or technological progress the production function is

$$Y = F[K.T(t), L] \quad \text{——— (25)}$$

Thus Solow neutral technological progress is capital augmenting.

[Note Consider Harrod-neutral technical progress.

$$\frac{KF_K}{LF_L} = \frac{K.r}{LW} = \frac{K/L}{W/r} = \frac{k}{\frac{MP_L}{MP_K}} = \frac{k}{\frac{MP_L(k)}{MP_K(k)}}$$

Constancy of $\frac{KF_K}{LF_L} \Rightarrow$ constancy of $\frac{k}{\frac{MP_L(k)}{MP_K(k)}}$

$\Rightarrow$ constancy of k

So Harrod neutrality $\Rightarrow$ constancy of k for a given capital-output ratio.

If we have constancy of k for an increase in capital-output ratio then it implies more capital is needed to maintain a given k in the production process. It means the technological progress is labour saving or capital using. On the other hand when we find constancy of k is maintained through a decrease in capital-output ratio then the technological progress is capital saving or labour using].

Labour-Augmenting Technological Progress (Harrod-neutral type) and the Solow Model

Suppose we consider only constant rates of technological progress. Then, in the neo-classical growth model with a constant rate of population growth, only labour-augmenting technological progress turns out to be consistent with the existence of a steady-state, that is, with constant growth rates of the various quantities in the long run.

We consider the production function as (with Harrod-neutral technological progress)

$$Y = F[K, L.T(t)] \rightarrow \text{as shown by equation (24)}$$

The condition for change in capital stock

$$\dot{K} = sF[K, L.T(t)] - \delta K \quad \text{——— (26)}$$

where the rate of technological progress is given by $\frac{\dot{T}(t)}{T(t)} = x$ ($x \Rightarrow$ constant rate of growth of $T(t)$)

We divide equation (26) by L and get

$$\dot{k} = sF[k, T(t)] - (n+\delta)k \quad \text{——— (27)}$$

[NTG]

$$k = \frac{K}{L}$$

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$$

$$= \frac{\dot{K}}{K} - n$$

$$\Rightarrow \dot{k} = \frac{\dot{K}}{K}.k - nk \Rightarrow \dot{k} = \frac{\dot{K}}{L} - nk$$

Now (26) ⇒

$$\dot{K} = sF[K, L.T(t)] - \delta K$$

$$\Rightarrow \frac{\dot{K}}{L} = s.\frac{1}{L}.F[K, L.T(t)] - \delta k$$

$$F[\lambda K, \lambda.L.T(t)] = \lambda.F[K, LT(t)]$$

Let $\lambda = \frac{1}{L}$. It ⇒

$$F\left[\frac{K}{L}, \frac{L}{L}T(t)\right] = \frac{1}{L}F[K, L.T(t)]$$

or, $$F[k, T(t)] = \frac{1}{L}F[K, L.T(t)]$$

Thus, $$\frac{\dot{K}}{L} = sF[k, T(t)] - \delta k$$

$$\therefore \dot{k} = sF[k, T(t)] - (n+\delta)k$$

(27) ⇒

$$\frac{\dot{k}}{k} = s\frac{F[k, T(t)]}{k} - (n+\delta) \qquad (28)$$

In equation (28), $\frac{\dot{k}}{k}$ equals the difference between two terms, where the first term is provided by the product of s and average product of capital, and the second term is $(n+\delta)$. The only difference is that now, for given k, the average product of capital, $\frac{F[k, T(t)]}{k}$, increases over time because of the growth in $T(t)$ at the rate x. In terms of figure 4, the downward sloping curve, $s\frac{F(\cdot)}{k}$, shifts continually to the right and hence, the level of k that corresponds to the intersection between this curve and the $(n+\delta)$ line also shifts continually to the right.

Equation (28) $\Rightarrow$

$$\frac{F[k, T(t)]}{k} = \frac{n+\delta}{s}$$

$$\Rightarrow \frac{F[k, T(t)]}{k} \text{ is constant}$$

$$\frac{F[k, T(t)]}{k} = \frac{1}{k} F[k, T(t)]$$

$$= F\left[1, \frac{T(t)}{k}\right]$$

due to CRS

$\Rightarrow F\left[1, \frac{T(t)}{k}\right]$ is constant only if $T(t)$ and k ~~grow~~ grow at the same rate

$$\Rightarrow \left(\frac{\dot{k}}{k}\right)^* = x$$

output per capita is given by

$$y = F[k, T(t)] = kF\left[1, \frac{T(t)}{k}\right]$$

Since $T(t)$ and k grow in the steady-state at the rate x, the steady-state growth rate of y equals x. Moreover, since $c = (1-s)y$, the steady-state growth rate of c also equals x.

In case of the production function

$$Y = F[K, L T(t)]$$

suppose we write ~~$\hat{L} = LT$~~ $\hat{L} = LT(t)$

$T(t)$ can be interpreted as a variable called efficiency of labour which is meant to reflect society's knowledge about production methods. As the available technology improves the efficiency of labour rises.

The term $L.T(t) = \hat{L}$ measures the number of effective workers. It implies number of workers multiplied by efficiency of each worker. Thus the technological progress is labour-augmenting and the rate of growth of technological progress is given by the constant term 'x'. Thus in the production function we have

$$\hat{k} = \frac{k}{T(t)} = \frac{K}{L.T(t)}$$ $\Rightarrow$ effective capital-labour ratio or capital per effective labour.

$$Y = F[K, L\,T(t)] = F[K, \hat{L}]$$

$$\Rightarrow \frac{Y}{\hat{L}} = F\left(\frac{K}{\hat{L}}, 1\right)$$

$$\Rightarrow \hat{y} = f(\hat{k}, 1) \qquad (29)$$

where output per unit of effective labour is $\hat{y} = \frac{Y}{\hat{L}}$

We have thus the dynamic equation for $\hat{k}$ as

$$\frac{\dot{\hat{k}}}{\hat{k}} = s\,\frac{f(\hat{k})}{\hat{k}} - (x + n + \delta) \qquad (30)$$

The only difference between equations (30) and (18), aside from the hats (^), is that the last term on the RHS of (30) includes the parameter x.

The term $(x+n+\delta)$ is now the effe[ctive]
depreciation rate for ~~k~~ the effect[ive]
capital-labour ratio $\hat{k} = \frac{K}{\hat{L}}$.

Here the steady-state rate of
growth of effective k, i.e. $\hat{k}$, is zer[o].
The steady-state value $\hat{k}^*$ satisf[ies]
the condition

$$s f(\hat{k}^*) = (x+n+\delta)\hat{k}^* \qquad (31)$$

The transitional dynamics of $\hat{k}$ are qualitat[ively]
similar to those of k in the previous mo[del]
as shown by figure 4. The case of transiti[onal]
dynamics of $\hat{k}$ is shown in terms of figu[re 6.]

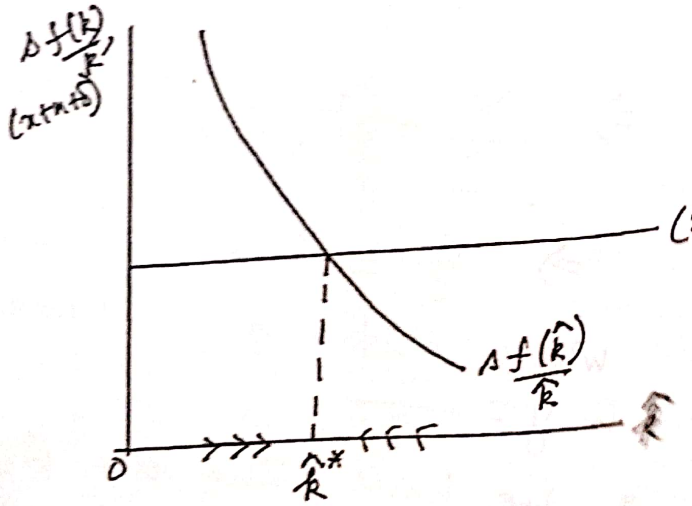

Figure 6

In figure 6 the downward sloping curve
$\frac{sf(\hat{k})}{\hat{k}}$ and the horizontal line is now ... cant
From figure 6 we find that the ...
is stable at steady-state.

At steady-state $\frac{\dot{\hat{k}}}{\hat{k}} = 0$ which implies $\frac{\dot{\hat{y}}}{\hat{y}} = 0$ and $\frac{\dot{\hat{c}}}{\hat{c}} = 0$. Thus at steady-state $\hat{k}$, $\hat{y}$ and $\hat{c}$ are constant

The variables K, Y and C grow accordingly in the steady-state at the rate $(n+x)$

We thus have the following table to explain steady-state growth rates in the ~~ASD~~ Solow model with technological progress

Variable	Steady-state growth rate
Capital per effective worker ($\hat{k}$)	0
output per effective worker ($\hat{y}$)	0
Consumption per effective worker ($\hat{c}$)	0
Capital per worker (k)	x
output per worker (y)	x
consumption per worker (c)	x
Capital (K)	$n+x$
Output (Y)	$n+x$
Consumption (C)	$n+x$

Technological Progress and Growth Accounting in Solow Model: Total Factor Productivity and Solow Residual

We consider the production function

$$Y = AF(K, L)$$

Where A is a measure of current level of technology which is called total factor productivity. Output increases not only because of increases in capital and labour but also because of increases in total factor productivity. Thus

$$\ln Y = \ln A + \ln F(K, L)$$

$$\Rightarrow \frac{dY}{Y} = \frac{dA}{A} + \frac{dF(K,L)}{F}$$

$$\text{or, } \frac{dY}{Y} = \frac{dA}{A} + \frac{\frac{\partial F}{\partial K} dK + \frac{\partial F}{\partial L} dL}{F}$$

$$\text{or, } \frac{dY}{Y} = \frac{dA}{A} + \frac{\left(\frac{\partial F}{\partial K}\right) \cdot K}{F} \frac{dK}{K} + \frac{\left(\frac{\partial F}{\partial L}\right) \cdot L}{F} \frac{dL}{L} \quad \text{——— (32)}$$

Equation (32) ⇒ growth in output consists of three components: growth ~~rate~~ of capital (weighted by share of capital) plus growth of labour (weighted by share of labour) plus growth in total ~~prod~~ factor productivity

In case of a Cobb-Douglas production function of the type

$$Y = A K^{\alpha} L^{1-\alpha}$$

we find equation (32) implies

$$\frac{dY}{Y} = \alpha \frac{dK}{K} + (1-\alpha)\frac{dL}{L} + \frac{dA}{A} \qquad (33)$$

Equation (33) can be rewritten as

$$\frac{dA}{A} = \frac{dY}{Y} - \left[\alpha \frac{dK}{K} + (1-\alpha)\frac{dL}{L}\right]$$

$\frac{dA}{A}$ is the change in output that cannot be explained by changes in inputs. Thus, the growth in total factor productivity is computed as a residual — that is, as the amount of ~~growth~~ output growth that remains after we have accounted for the determinants of growth that we can measure.

Thus $\frac{dA}{A}$ is known as the <u>Solow residual</u>, after Robert Solow, who first showed how to compute it.