Linear independence and dependence of vectors

Let V be a vector space over a field F. If $\alpha_1, \alpha_2, \dots \alpha_n \in V$, then any vector α is said to be a linear combination of the vectors $\alpha_1, \alpha_2, \dots \alpha_n$ if

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$$ where the scalars $a_1, a_2, \dots, a_n \in F$

Ex: Express $(3,4,5)$ as a linear combination of $\alpha = (1,2,3)$, $\beta = (2,3,4)$, and $\gamma = (4,3,2)$ in the vector space V_3 of real numbers.

Let $(3,4,5) = c_1(1,2,3) + c_2(2,3,4) + c_3(4,3,3)$

$c_1, c_2, c_3 \in R$

$\therefore (3,4,5) = (c_1 + 2c_2 + 4c_3,\ 2c_1 + 3c_2 + 3c_3,\ 3c_1 + 4c_2 + 3c_3)$

$\therefore$

$$c_1 + 2c_2 + 4c_3 = 3 \quad \text{--- (1)}$$
$$2c_1 + 3c_2 + 3c_3 = 4 \quad \text{--- (2)}$$
$$\cancel{4}\,3c_1 + 4c_2 + 3c_3 = 5 \quad \text{--- (3)}$$

from (1) & (2)

$$2c_1 + 4c_2 + 8c_3 = 6$$
$$\underline{2c_1 + 3c_2 + 3c_3 = 4}$$
$$c_2 + 5c_3 = \cancel{10}\,2 \quad \text{--- (4)}$$

from (1) & (3)

$$3c_1 + 6c_2 + 12c_3 = 9$$
$$\underline{3c_1 + 4c_2 + 3c_3 = 5}$$
$$2c_2 + \cancel{9}10c_3 = 4$$
$$2c_2 + 5c_3 = 2$$

$$\Delta = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 3 \\ 3 & 4 & 3 \end{vmatrix}$$

$$= 1\cancel{(6-12) + 2(9-4) + 4(8-9) = -6+10-4} = 0$$

$$2c_2 + 9c_3 = 4$$
$$2c_2 + 10c_3 = 4$$

$\therefore c_3 = 0$

$c_2 = 2$

$c_1 = 3 - 4$
$= -1$

$(3,4,5) = (-1)(1,2,3) + 2(2,3,4) + 0(4,3,3)$.

~~Linear Independence and dependence of vectors.~~

~~Let V be a vector space over a field F.~~
~~If $\alpha_1, \alpha_2, \cdots, \alpha_n \in V$, then any vector α is said to be a linear combination of the vectors $\alpha_1, \alpha_2, \cdots, \alpha_n$ if $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$,~~
~~where the scalars $a_1, a_2, \cdots, a_n \in F$.~~

~~Let~~ ~~Ex~~

Linear Independence and dependence of vectors

Let V be a vector space over a field F. A finite set of vectors $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of vectors of V is said to be linearly dependent, if ~~every relation~~ there exists scalars $c_1, c_2, \cdots, c_n$ not all zero in F such that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0 \quad - (i)$$

The set of vectors $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is said to be linearly independent in V if every relation of the form

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0 \qquad c_i \in F,\ 1 \le i \le n$$

implies $c_i = 0$ for each $1 \le i \le n$.

- A null set is assumed to be linearly independent.
- A system consisting of a single non-zero vector is always linearly independent.

Proof Let $S = \{\alpha\}$, $\alpha \neq 0$, $S \subset V$, $a \in F$

$$a\alpha = 0 \Rightarrow a = 0$$

$\therefore$ S is linearly independent.

- Every superset of a linearly dependent set of vectors is linearly dependent.
- Any subset of a linearly independent set of vectors is linearly independent.

• Let V be a vector space over the field F. Then the set S of non-zero vectors $\alpha_1, \alpha_2, \dots, \alpha_n$ is linearly dependent, if and only if some elements of S be a linear combination of the others.

• Let V be a vector space over a field F. The non zero vectors $\alpha_1, \alpha_2, \dots, \alpha_n \in V$ are linearly dependent, if and only if one of them be a linear combination of the preceding vectors.

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Ex ⓐ Show that the vectors (1, 2, 3) and (4, -2, 7) are linearly independent in V_3 over the field F of real numbers.

ⓑ If the vectors (0, 1, a), (1, a, 1), (a, 1, 0) of the vectors space $R^3(R)$ be linearly dependent, then find the value of a.

Solution

ⓐ

$$c_1(1,2,3) + c_2(4,-2,7) = 0 \qquad c_1, c_2 \in R$$
$$= (0,0,0)$$

$\therefore$ $c_1 + 4c_2 = 0$. ①

$2c_1 - 2c_2 = 0$. ②

$3c_1 + 7c_2 = 0$. ③.

from ① & ② . $c_1 = c_2 = 0$, which satisfies ③.

$\therefore$ The vectors are linearly independent.

(b) Let $c_1, c_2, c_3 \in \mathbb{R}$

$$c_1(0,1,a) + c_2(1,a,1) + c_3(a,1,0) = 0 = (0,0,0)$$

$\Rightarrow$ $c_2 + c_3 a = 0$ ... (1)

$c_1 + ac_2 + c_3 = 0$ ... (2)

$ac_1 + c_2 + 0.c_3 = 0$ ... (3)

from (1) & ~~(3)~~ $\Rightarrow$ $\frac{c_2}{a} = \frac{c_3}{-1} = k$.

$\therefore$ $c_2 = ka$

$c_3 = -k$.

subst. in (2) $\Rightarrow$ $c_1 + ka^2 - k = 0$

$c_1 = k(1-a^2)$

subst. in (3)

$ak(1-a^2) + ka = 0$

$\because k \neq 0$ $\quad a(1-a^2) + a = 0$

$\Rightarrow$ $2a - a^3 = 0$

$\Rightarrow$ $a(2-a^2) = 0$

$\therefore$ either $a = 0$ or $a^2 = 2$

$a = \pm\sqrt{2}$

If $k = 0$, then $c_1 = c_2 = c_3 = 0$ so that the given vectors ~~become~~ are linearly independent. $\therefore$ $k \neq 0$.

$\therefore$ $a = 0, +\sqrt{2}, -\sqrt{2}$

Ex Write the vector $\alpha = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of

$\alpha_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ $\alpha_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$

$\Rightarrow$ $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = c_1\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + c_2\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$$c_1 + c_2 + c_3 = 3 \quad \text{--- (1)}$$
$$c_1 + c_2 - c_3 = -1 \quad \text{--- (2)}$$
$$-c_2 = 1 \quad \text{--- (3)}$$
$$-c_1 = -2 \quad \text{--- (4)}$$

$\rightarrow c_1 + c_2 = 1$

$c_1 = 2$

$c_2 = -1$

$c_3 = 2$

from (1), (3) & (4)

$\cancel{c_3 = 6}$

$\therefore \alpha = 2\alpha_1 - \alpha_2 + 2\alpha_3$ Ans.

Ex Show that $\alpha_1 = (1, 5, 2)$
$\alpha_2 = (1, 1, 0)$
$\alpha_3 = (0, 0, 1)$ are linearly independent vectors.

Soln

Let c_1, c_2, c_3 be scalars such that

$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta = (0, 0, 0)$$

$$\Rightarrow c_1(1, 5, 2) + c_2(1, 1, 0) + c_3(0, 0, 1) = (0, 0, 0)$$

$$\Rightarrow c_1 + c_2 + 0.c_3 = 0$$
$$5c_1 + c_2 + 0.c_3 = 0$$
$$2c_1 + 0.c_2 + 1.c_3 = 0$$

The co-eff. matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 5 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$|A| = -4 \neq 0.$

$\therefore r(A) = 3 =$ the number of unknowns.

$\therefore c_1 = c_2 = c_3 = 0$

The vectors are linearly independent.

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Ex Show that the vectors $(-1,2,1)$, $(3,0,-1)$ and $(-5,4,3)$ are linearly dependent in $V_3(R)$.

Let c_1, c_2, c_3 are scalars such that
$$c_1(-1,2,1) + c_2(3,0,-1) + c_3(-5,4,3) = 0 = (0,0,0)$$

$\Rightarrow$ $-c_1 + 3c_2 - 5c_3 = 0$ --- ①
$2c_1 + 0.c_2 + 4c_3 = 0$ --- ②
$c_1 - c_2 + 3c_3 = 0$ --- ③.

[~~Remember~~ these ~~equations~~ will have a non zero solution in V_3, if the rank of the coefficient matrix be less than 3, the number of unknown c_1, c_2, c_3. But if the rank be 3 the only solution will be $c_1 = c_2 = c_3 = 0$].

$$A = \begin{bmatrix} -1 & 3 & -5 \\ 2 & 0 & 4 \\ 1 & -1 & 3 \end{bmatrix}. \qquad |A| = -1\times(4) + 3(4-6) - 5(-2) = -4-6+10 = 0.$$

$\therefore$ rank (A) is less than 3, the number of unknowns. Hence the vectors are linearly dependent.

Adding ① & ③ $\quad 2c_2 - 2c_3 = 0$
$\Rightarrow c_2 = c_3$.

~~Subst. in ②~~ from ③
$$c_1 = -2c_3.$$
$$\therefore \frac{c_1}{-2} = \frac{c_2}{1} = \frac{c_3}{1} = k = 1.$$

$\therefore c_1 = -2,\ c_2 = 1,\ c_3 = 1$. is a non-zero solution

H.W ~~Submit~~ Submit on 1/4/03
at 9-00 a.m.

1. Express the vector ~~Show that~~ $\alpha = (8, 17, 36)$ as a linear combination of $\alpha_1 = (1, 0, 5)$, $\alpha_2 = (0, 3, 4)$ and $\alpha_3 = (1, 1, 1)$.

2. Show that the vectors $(1, 0, -1)$, $(2, 1, 3)$, $(-1, 0, 0)$, $(1, 0, 1)$ are linearly dependent.

3. Show that $(1, 2, 1)$, $(2, 1, 1)$, $(1, 1, 2)$ are linearly independent.

4. Express $(2, 3, 4)$ as a linear combination of $(1, 1, 1)$ and $(-1, 0, 1)$.

5. Determine k so that $(1, 3, 1)$, $(2, k, 0)$ and $(0, 4, 1)$ are linearly dependent.