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STABLE AND UNSTABLE EQUILIBRIUM

Defn.

Stable Equilibrium – A body is said to be in stable equilibrium if, after it is slightly displaced from its position of equilibrium, it has a tendency to return to its original position.

example – A book is lying on a horizontal surface. Because, if the book is lifted from one edge and then allowed to fall, it will come back to its original position.

Another example, the pendulum of a clock. Because, if it is pulled aside and then released; it will move to and fro and ultimately come to rest in its original position.

Unstable Equilibrium – A body is said to be in unstable equilibrium when, if it be slightly displaced, the force tend to move it still further from its position of equilibrium.

example – A bottle standing on the edge of its mouth.

Because, it does not regain its original position after being slightly disturbed by an external force.

Another example, a cone resting on its apex.

Neutral Equilibrium. A body is said to be in neutral equilibrium, when if it be slightly displaced from its position of equilibrium, it remains in equilibrium in the new position (ie displaced position).

example – A uniform sphere resting on a horizontal table or a cone resting with its slant side in contact with the horizontal plane.

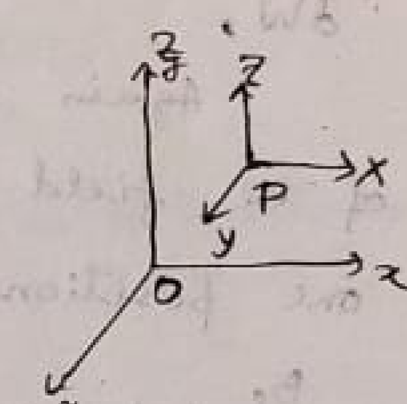
Field of force

If a system of forces act in a space S, so that at every point of the space there acts a definite resultant force, whose magnitude and direction is known, then such a space is called a Field of force or simply a Field and the forces of the system are called forces of the field.

Conservative field of force

Suppose a system of bodies, connected together and subject to certain constraints. If the system of bodies occupy a certain position in a given field of force and upon receiving a small displacement moves to a neighbouring position, such that none of the constraints are violated. Let us calculate the work done by the forces of the field which act at different points of the system.

Let Ox, Oy, Oz be a system of rectangular axes chosen in the field and let X, Y, Z be the components parallel to the axes of the force P which acts at the point (x, y, z) of the system.



As the field is given X, Y, Z are known functions of x, y, z. The work done by P due to small displacement is $X\,dx + Y\,dy + Z\,dx$.

The work done by the force P due to small displacement is $X\,dx + Y\,dy + Z\,dz$.

Therefore the total work done by the forces of the field which act on the system $= \Sigma\,(X\,dx + Y\,dy + Z\,dz)$ ---- (i)

Let the co-ordinates of the system of bodies be $\theta_1, \theta_2, \ldots \theta_m$. Then x, y, z and X, Y, Z are all known functions of $\theta_1, \theta_2, \ldots \theta_m$.

Denoting the small work done by dW, we have from (i)

$$dW = f_1\, d\theta_1 + f_2\, d\theta_2 + \cdots + f_m\, d\theta_m \quad \text{---(ii)}$$

where $f_1, f_2, \ldots f_m$ are all known functions of the co-ordinates $\theta_1, \theta_2, \ldots \theta_m$.

If there exists a function ϕ of $\theta_1, \theta_2, \ldots \theta_m$ such that $f_1 = \dfrac{\partial \phi}{\partial \theta_1}$, $f_2 = \dfrac{\partial \phi}{\partial \theta_2}$, ... then we have from (ii)

$$dW = \frac{\partial \phi}{\partial \theta_1} d\theta_1 + \frac{\partial \phi}{\partial \theta_2} d\theta_2 + \cdots + \frac{\partial \phi}{\partial \theta_m} d\theta_m$$

$$\text{ie} \quad dW = d\phi \quad \text{---(iii)}$$

$\therefore$ Integrating, $W = \phi + C$ where C is a constant. ---(iv)

W is called the work function.

In reality, using (ii) and (iii) in (i) we see that $(X\,dx + Y\,dy + Z\,dz)$ becomes an exact differential dW.

Again using (iii), the work done by the forces of the field when the system of bodies moves from one position A to another position B will be equal to

$$\int_A^B d\phi = \phi_B - \phi_A \quad \text{---(v)}$$

where ϕ_A denotes the value of ϕ for the position A.

$$\therefore \phi_A = \phi(\alpha_1, \alpha_2, \ldots \alpha_m)$$

if $\theta_1 = \alpha_1$, $\theta_2 = \alpha_2$, ... define the position A. Similarly for ϕ_B.

From (v), it is seen that the work done depends only on the initial and final positions of the system and not on the route taken by the system of bodies to move from one position to the other provided no constraints are violated.

In such a case the field of force is called conservative and the forces of the field are said to form a conservative system.

Potential energy of a system

Suppose that we have conservative field of force and suppose there is a standard position D, defined by $\theta_1 = \beta_1$, $\theta_2 = \beta_2$ ...

Then the work done by the forces of the field when the system moves from its present position to the standard position is $\phi(\beta_1, \beta_2, \dots) - \phi(\theta_1, \theta_2, \dots)$

This work done depends only on the co-ordinates of the present position and is called the potential energy of the system in the position $(\theta_1, \theta_2, \dots \theta_m)$.

If this potential energy is denoted by V, we get

$V = \int_P^D d\phi$ where P is the present position $(\theta_1, \theta_2, \dots)$

$= \phi_D - \phi_P$

$\therefore dV = d\phi_D - d\phi_P$

Now D is the standard position giving $d\phi_D = 0$.

$\therefore dV = -d\phi_P$

ie $dV = -d\phi$, present position can vary.

ie $dV = -d\phi = -dW$ [From eqn. (iv) of previous article]

$\therefore dW + dV = 0$

or, $W + V = \text{constant}.$

Thus when the system moves in such a way that the forces of the field do positive work, the potential energy of the system decreases by the amount of work done.

It is proved in Dynamics that when a system moves from one position to another, the change in its Kinetic Energy is equal to the work done by the external forces.

Therefore if K denotes the Kinetic energy of the system then $dK = dW = -dV$.

$\therefore dK + dV = 0$

Thus in a conservative field, the sum of the Kinetic and Potential energies of a system is constant. i.e. the same at all positions.

All natural fields of force are conservative. As such, the simplest example of a conservative field is the field due to gravity.

The energy Test of Stability.

Suppose a body or a system of bodies, connected together and subject to certain constraints, is in equilibrium under the action of forces in a conservative field. By the principle of virtual work, no work is done by the forces of the field when the system receives any arbitrary small virtual displacement consistent with its geometrical conditions. As no work is done, there is no change in the Kinetic energy of the system and no change in its potential energy as in a conservative field, the sum of the Kinetic and Potential energies of a system is constant.

Conversely, if there is no change in the potential energy of the system for any arbitrary small displacement, then there is no change in its Kinetic energy and so no work is done by the forces of the field and therefore, by the converse of the principle of virtual work, the system is in equilibrium.

Thus for positions of equilibrium we have $dV = 0$, where V denotes the potential energy, and therefore V is a maximum or a minimum or stationary at the positions of equilibrium.

(6) First, let us consider a position of equilibrium of the system for which V is maximum.
Let the system be displaced to a neighbouring position and then set free. In the displaced position the value of V is decreased.
When set free, it begins to move under the action of the forces of the field and therefore kinetic energy increases and so potential energy decreases.
Therefore it cannot move back to its original position of equilibrium and the equilibrium is unstable.

Next, we consider a position of equilibrium of the system is one in which V is a minimum and the system is displaced slightly and then set free.
Then V is increased in the displaced position and when set free the system begins to move, and it must so move so that its potential energy decreases, for by moving, it is gaining kinetic energy.
Therefore the system tends to move back to its original position of equilibrium, and the equilibrium is stable.

Lastly, when V is stationary in the position of equilibrium, it is increased for some displacements and decreased for some others. Therefore from the proceeding arguments it follows that the equilibrium is stable for some displacements and unstable for others and therefore the equilibrium is on the whole unstable.

Hence we have the theorem known as the Energy test of Stability:

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The positions of equilibrium of a system in a conservative field are those positions in which the potential energy of the system is a maximum or a minimum or stationary.

Positions of maximum ~~poten~~ potential energy are positions of unstable equilibrium, positions of minimum potential energy are positions of stable equilibrium, and the positions in which the potential energy is stationary are on the whole unstable.

For the discussion above, the assumption has been that none of the neighbouring positions is also a position of equilibrium. But if there be such a neighbouring position, then the equilibrium is neutral for the displacement to that position.

(8) Energy test of stability for a case in which gravity is the only external force acting on the system.

Let a body or system of bodies, connected together and subject to certain constraints, be under the action of no forces except their weights and the forces of constraints, and the forces of constraints are supposed to be such that they do not appear in the equation of virtual work.

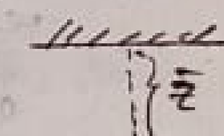
Let $w_1, w_2, \ldots$ be the weights of different bodies of the system and let $z_1, z_2, \ldots$ be the depth of their centres of gravity below some fixed horizontal plane.

Let W be the total weight of the system,

$\bar{z}$ = the depth of its c.g below the fixed plane

& V = its potential energy.

The virtual work of the weights for a small displacement is



$$-\delta V = w_1 \delta z_1 + w_2 \delta z_2 + \ldots$$

Integrating, $-V + C = w_1 z_1 + w_2 z_2 + \ldots$

$$= W\bar{z} \qquad \left[\because \bar{z} = \frac{w_1 z_1 + w_2 z_2 + \ldots}{w_1 + w_2 + \ldots} \text{ and } W = w_1 + w_2 + \ldots \right]$$

or $V = C - W\bar{z}$, where C is a constant.

$\therefore$ From this relation we see that, when $\bar{z}$ is a maximum, V is minimum. $\Rightarrow$ The equilibrium is stable

& When $\bar{z}$ is a minimum, V is maximum $\Rightarrow$ The equilibrium is unstable.

Thus the positions of equilibrium of the system are those in which the depth of its c.g below a fixed horizontal plane is a maximum or a minimum and the equilibrium is stable in the former case and unstable in the later.

(9)

Stability of the system having one degree of freedom

Let the system have one degree of freedom, then its position is determined by a single co-ordinate, say θ. Hence to determine the positions of equilibrium and stability, we express the depth of the c.g in terms of the variable θ, the angle through which the body is tilted from its position of equilibrium.

From the geometry of the equation, $\bar{z}$ can be expressed in terms of θ.

ie $$\bar{z} = f(\theta).$$

The positions of equilibrium of the system are those in which depth of its c.g below a fixed horizontal plane is a maximum or a minimum and the equilibrium is stable in the former case and unstable in the later.

We know from the previous discussion, if $\bar{z}$ is depth of c.g, then $\bar{z}$ is min $\Rightarrow$ equilibrium is unstable & $\bar{z}$ is max $\Rightarrow$ equilibrium is stable

The required value of θ are given by

$$\frac{d\bar{z}}{d\theta} = 0 \quad \text{or,} \quad f'(\theta) = 0 \quad \text{as} \quad \bar{z} = f(\theta)$$

By solving the eqn. $f'(\theta) = 0$ we get the values of θ equal to α, β - - - - (say)

If at $\theta = \alpha$, $f'(\theta) = 0$ and $f''(\theta) > 0$, then $\bar{z}$ is minimum and the position is of unstable equilibrium.

If at $\theta = \alpha$, $f'(\theta) = 0$ and $f''(\theta) < 0$, then $\bar{z}$ is maximum, This gives position is of stable equilibrium.

If $f''(\alpha) = 0$, we must find the higher derivatives of $\bar{z}$ and decide, according to the rules of Calculus, whether $\theta = \alpha$ makes $\bar{z}$ a maximum or a minimum. Similarly the other values of θ are to be tested for stability.

Condition of stability of equilibrium of a perfectly rough heavy body resting on a fixed body. (ie to find whether the equilibrium is stable or unstable)

To discuss this problem, first we will make some assumptions which are given by the following:

i) To solve the problem, it is necessary to consider displacements which are to be given to the body.
We assume that these very small displacements take place in a vertical plane of symmetry so that the centre of gravity of the upper body moves in this vertical plane during its motion.

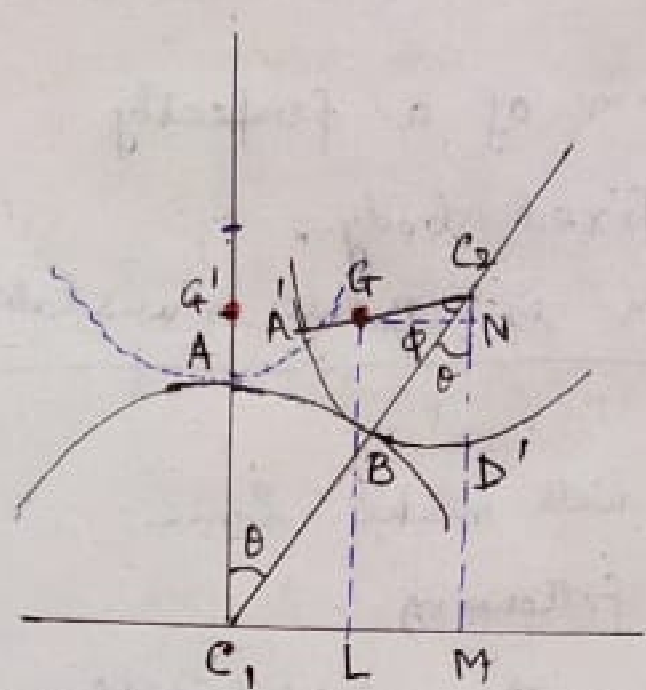
ii) During the displacements, the parts of the two bodies which are in contact are portions of spheres. As such, their sections by the above mentioned vertical plane of symmetry are two circular arcs and during the displacements the upper circular arc rolls on the lower one. No sliding is possible since, the bodies are perfectly rough, by assumption.

Now for the problem, two different parts need to be considered.

(A) We first assume that the surface of the lower body, which is fixed, is convex upwards and the upper body rests in equilibrium at the highest point of the fixed body.

(B) Secondly, we assume that the surface of the lower fixed body is concave upwards and upper body rests in equilibrium at the lowest point of the concave surface.

We solve for (A) first



The upper body is kept in equilibrium by two forces acting on it, viz, its weight acting vertically downwards through its centre of gravity and the total reaction at the point of contact; it follows that the c.g is on the vertical through the point of contact.

Let h be its height above A, the point of contact in the position of equilibrium. ie $AG' = h$.

Let r_1 and r_2 be the radii of the circular arcs in which the vertical plane of symmetry cuts the lower and the upper body respectively, the two arcs touching at A.

Let $A'BD'$ be the slightly displaced position of the upper circular arc, C_2 being the new position of its centre, B the new point of contact, A' the new position of A and G the new position of the c.g of the upper body on $A'C_2$ where $A'G = h$.

Since the upper body rolls over the lower, arc AB = arc $A'B$.

Let C_1 be the centre of the lower circular arc, then C_1BC_2 is one straight line and

$C_1B = r_1$, $C_2B = A'C_2 = r_2$.

$C_2G = C_2A' - GA' = r_2 - h$ or, $h - r_2$

according as $A'G \lessgtr A'C_2$

$A'G > A'C_2$ means the c.g of the upper sphere is above ...

Let $\angle AC_1B = \angle BC_2D' = \theta$

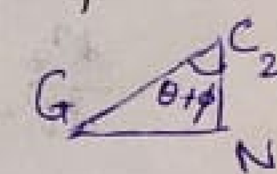
$\angle A'C_2B = \phi$, θ and ϕ being very small angles.

Then arc $AB = r_1\theta$ & arc $A'B = r_2\phi$

But arc $AB =$ arc $A'B$

$$\therefore \quad r_1\theta = r_2\phi \qquad \text{---(i)}$$

Let $z =$ height of G in the new position above the horizontal through $C_1 = GL$



$$\therefore \quad z = NM = C_2M - C_2N$$

$$= C_1C_2\cos\theta - (r_2-h)\cos(\theta+\phi)$$

$$= (r_1+r_2)\cos\theta - (r_2-h)\cos\left(\theta + \frac{r_1\theta}{r_2}\right)$$

[from (i)]

$$= (r_1+r_2)\cos\theta - (r_2-h)\cos\left(\frac{r_1+r_2}{r_2}\theta\right)$$

Differentiating

$$\therefore \frac{dz}{d\theta} = -(r_1+r_2)\sin\theta + (r_2-h)\frac{r_1+r_2}{r_2}\sin\left(\frac{r_1+r_2}{r_2}\theta\right)$$

$$\therefore \frac{d^2z}{d\theta^2} = -(r_1+r_2)\cos\theta + (r_2-h)\left(\frac{r_1+r_2}{r_2}\right)^2\cos\left(\frac{r_1+r_2}{r_2}\theta\right) \qquad \text{---(2)}$$

In the position of equilibrium, we have $\theta = 0$ and from above we see that $\frac{dz}{d\theta} = 0$ when $\theta = 0$.

Here z is the height of the c.g of the of the upper body, the equilibrium is stable as the value of z, when $\theta = 0$ is a minimum.

& unstable if z is a maximum.

ie according as $\frac{d^2z}{d\theta^2}$ is $>$ or <0 when $\theta = 0$

Now $\left[\frac{d^2z}{d\theta^2}\right]$ at $\theta = 0$ is equal to

$$-(r_1+r_2) + (r_2-h)\left(\frac{r_1+r_2}{r_2}\right)^2$$

$$= (r_1+r_2)\left\{-1 + \frac{(r_2-h)(r_1+r_2)}{r_2^2}\right\}$$

$$= \left(\frac{r_1+r_2}{r_2}\right)^2\left\{(r_2-h) - \frac{r_2^2}{r_1+r_2}\right\}$$

$$= \left(\frac{r_1+r_2}{r_2}\right)^2\left\{\frac{r_1r_2}{r_1+r_2} - h\right\}$$

Hence the equilibrium will be stable when

$$\frac{d^2z}{d\theta^2} > 0 \quad \text{ie} \quad \text{when} \quad \frac{r_1 r_2}{r_1 + r_2} - h > 0$$

ie when $\frac{r_1 r_2}{r_1 + r_2} > h$. ie when $\frac{1}{h} > \frac{1}{r_1} + \frac{1}{r_2}$

& the equilibrium will be unstable when

$$\frac{d^2z}{d\theta^2} < 0 \quad \text{ie} \quad \text{when} \quad \frac{r_1 r_2}{r_1 + r_2} - h < 0$$

ie when $\frac{r_1 r_2}{r_1 + r_2} < h$ ie when $\frac{1}{h} < \frac{1}{r_1} + \frac{1}{r_2}$

But if $\frac{1}{h} = \frac{1}{r_1} + \frac{1}{r_2}$, then the higher derivatives of z need to be considered.

$$\frac{d^3z}{d\theta^3} = (r_1 + r_2)\left[\sin\theta - \frac{r_1 + r_2}{r_2}\sin\left(\frac{r_1 + r_2}{r_2}\theta\right)\right]$$

which vanishes when $\theta = 0$

Now $$\frac{d^4z}{d\theta^4} = (r_1 + r_2)\left[\cos\theta - \left(\frac{r_1 + r_2}{r_2}\right)^2 \cos\left(\frac{r_1 + r_2}{r_2}\theta\right)\right]$$

Again when $\theta = 0$.

$$\frac{d^4z}{d\theta^4} = (r_1 + r_2)\left[1 - \left(\frac{r_1 + r_2}{r_2}\right)^2\right]$$

$$\therefore \frac{d^4z}{d\theta^4} \gtrless 0 \Rightarrow 1 - \left(\frac{r_1 + r_2}{r_2}\right)^2 \gtrless 0$$

But $1 - \left(\frac{r_1 + r_2}{r_2}\right)^2 < 0$ $\quad \therefore \frac{d^4z}{d\theta^3} < 0$ when $\theta = 0$

Thus in this case, z is maximum and the equilibrium is unstable.

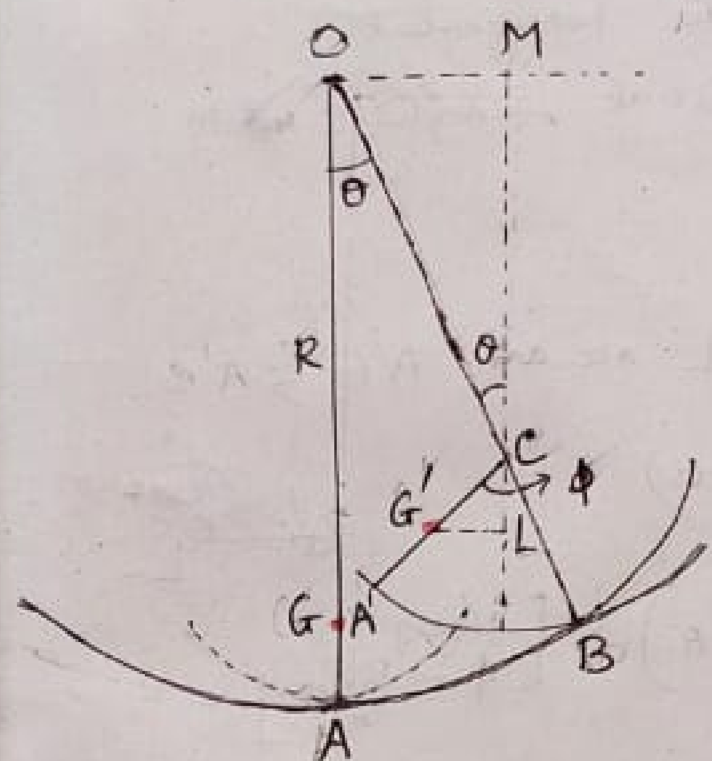
Hence the equilibrium is stable only if

$$\frac{1}{h} > \frac{1}{r_1} + \frac{1}{r_2}$$

and unstable in all the other cases.

We next solve for (B)

ie We assume that the surface of the lower fixed body is concave upwards and the upper body rests in equilibrium at the lowest point of the concave surface.



As before the upper body is kept in equilibrium by the action of two forces on it, viz, its weight acting vertically downwards through its centre of gravity and the total reaction at the point of contact.

Since the two forces balance, hence the centre of gravity of the body must be on the vertical through the point of contact.

Let A be the point of contact in the equilibrium position and h be the height of G (c.g of the upper body) above A.

ie $AG = h$.

Let r, R be the radii of the circular arcs in which the vertical plane of symmetry cuts the upper and the lower body respectively, the two arcs touching at A.

After receiving the small displacement, let $A'B$ be the slightly displaced position of the upper circular arc, C being the new position of its centre, B the new point of contact, A' the new position of A and G' the new position of G on $A'C$ where $A'G' = h$.

Since the upper body rolls on the lower, arc AB = arc $A'B$

Let O be the centre of the lower circular arc, then OCB is one straight line and $OB = R$,

$CB = CA' = r < R$

and ~~CG'~~ and $CG' = (r-h)$ or $(h-r)$ according as $A'G' \lessgtr A'C$.

(15)

Let $\angle AOB = \theta$ and $\angle A'CB = \phi$,

where θ, ϕ are very small angles.

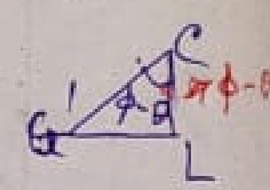
We have $R\theta = \text{arc } AB = \text{arc } A'B = r\phi$ --- (1)

Let z be the depth of G' below the horizontal through O, which is cut by the vertical through C ~~and M~~ and M.

Draw $G'L \perp$ on MC

Then $z = ML = MC + CL$ or $MC - CL$ acc. as $A'G' \gtrless A'C$

$$= (R-r)\cos\theta + (r-h)\cos(\phi-\theta)$$



$$= (R-r)\cos\theta + (r-h)\cos\left(\frac{R-r}{r}\theta\right) \quad \left[\begin{array}{l}\text{Using (1)} \\ \phi = \frac{R}{r}\theta\end{array}\right]$$

Differentiating we get

$$\frac{dz}{d\theta} = -(R-r)\sin\theta - (r-h)\left(\frac{R-r}{r}\right)\sin\left(\frac{R-r}{r}\cdot\theta\right)$$

Again differentiating

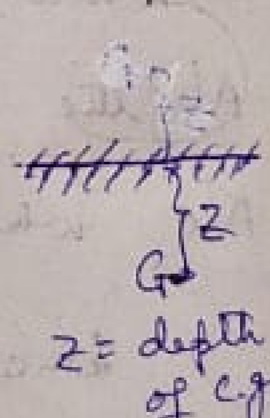
$$\frac{d^2z}{d\theta^2} = -(R-r)\cos\theta - (r-h)\left(\frac{R-r}{r}\right)^2\cos\left(\frac{R-r}{r}\cdot\theta\right) \quad \text{--- (2)}$$

In the position of equilibrium, we have $\theta = 0$ and from above we see that $\frac{dz}{d\theta} = 0$ when $\theta = 0$

Now, the weight being constant in the given gravitational field, the potential energy depends on z.

Therefore when $\theta = 0$, the equilibrium is unstable or stable as z is a minimum or maximum.

ie according as $\frac{d^2z}{d\theta^2} >$ or < 0



First when $\theta = 0$, we have

$$\frac{d^2z}{d\theta^2} = -(R-r) - (r-h)\left(\frac{R-r}{r}\right)^2$$

So for stable equilibrium

$$-(R-r) - (r-h)\left(\frac{R-r}{r}\right)^2 < 0$$

or, $$1 + (r+h)\frac{(R-r)}{r^2} > 0$$

$$\therefore \frac{d^2z}{d\theta^2} \gtrless 0 \Rightarrow$$

$$-(R-r) - (r-h)\left(\frac{R-r}{r}\right)^2 \geq 0$$

or, $$-1 - (r-h)\frac{(R-r)}{r^2} \geq 0$$

or, $$1 + (r-h)\frac{(R-r)}{r^2} \leq 0$$

or, $$r^2 + (r-h)(R-r) \leq 0$$

or, $$r^2 + Rr - Rh - r^2 + rh \leq 0$$

or, $$Rr - h(R-r) \leq 0$$

or, $$h(R-r) - Rr \geq 0$$

or, $$h(R-r) \geq Rr.$$

ie $$\frac{1}{h} \leq \frac{R-r}{Rr} \quad \text{ie} \quad \frac{1}{h} \leq \frac{1}{r} - \frac{1}{R} \quad \cdots \quad (3)$$

Again, if $\frac{d^2z}{d\theta^2} = 0$ when $\theta = 0$ ie if $\frac{1}{h} = \frac{1}{r} - \frac{1}{R}$

ie $Rr = h(R-r)$

then the higher derivatives of z to be calculated.

When $h = \frac{Rr}{(R-r)}$, from (2) we get

$$\frac{d^2z}{d\theta^2} = -(R-r)\cos\theta - \left[r - \frac{Rr}{R-r}\right]\left(\frac{R-r}{r}\right)^2 \cos\left(\frac{R-r}{r}\cdot\theta\right)$$

$$= -(R-r)\cos\theta + \frac{r^2}{(R-r)}\left(\frac{R-r}{r}\right)^2 \cos\left(\frac{R-r}{r}\cdot\theta\right)$$

$$= (R-r)\left[-\cos\theta + \cos\left(\frac{R-r}{r}\right)\theta\right]$$

$\therefore$ Differentiating

$$\frac{d^3z}{d\theta^3} = (R-r)\left[\sin\theta - \left(\frac{R-r}{r}\right)\sin\left(\frac{R-r}{r}\right)\theta\right]$$

But $\left[\frac{d^3z}{d\theta^3}\right]_0 = 0$

$$\therefore \frac{d^4z}{d\theta^4} = (R-r)\left[\cos\theta - \left(\frac{R-r}{r}\right)^2 \cos\left(\frac{R-r}{r}\right)\theta\right]$$

Now $$\left(\frac{d^4z}{d\theta^4}\right)_{\theta=0} = (R-r)\left[1 - \left(\frac{R-r}{r}\right)^2\right] = (R-r)R\frac{(2r-R)}{r}.$$

$$\therefore \frac{d^4z}{d\theta^4} \gtrless 0 \Rightarrow \left[1 - \left(\frac{R-r}{r}\right)^2 \gtrless 0 \text{ ie } \left(\frac{R-r}{r}\right)^2 \lessgtr 1 \text{ ie } R-r \lessgtr r\right] \text{ ie } R \lessgtr 2r.$$

(17) i.e. $\frac{d^4z}{d\theta^4} \gtrless 0$ if $R \lessgtr 2r$.

$\therefore$ In this case, z is minimum or a maximum and therefore, the equilibrium is unstable or stable according as $R \lessgtr 2r$.

But when $h = \frac{Rr}{(R-r)}$ and $R = 2r$,

then we get $h = 2r$ and also

$$z = (2r - r)\cos\theta + (r - 2r)\cos\left(\frac{2r-r}{r}\right)\theta$$

$$= r\cos\theta - r\cos\theta = 0 \quad \text{for all } \theta.$$

$\therefore$ $\frac{dz}{d\theta} = 0$ for all θ.

Thus every displaced position is also a position of equilibrium and the equilibrium $\theta = 0$ is really neutral.