Th Prove that x is an interior pt of a set S iff there exists a rational number r such that $]x-r, x+r[\subset S$.

*[If $p \in S$, then p is an adherent point ~~...~~ pt of S, since each nbd of p contains p which belongs to S. If S' be the derived set of S and if $p \in S'$, then p is a limit point of S and so each nbd of p contains a point of S other than p. Hence p is an adherent point of S.

p is an adherent pt of $S \Leftrightarrow p \in S$ or $p \in S'$] Read carefully

~~Thus~~

- Adherent Point (Or a contact Point) and Closure of a set

A point p is called an adherent point of a set S if every nbd of p contains a point of S. $\Rightarrow$ each nbd of p $N(p) \cap S \neq \phi$.

or $]p-\epsilon, p+\epsilon[$ contains a pt of S.

Important

[Each point of S and each limit of S will be adherent pts ~~of S~~, or Each adherent point of S need not be always a limit of S] *

Closure of a set - The set of all adherent pt of a set is called the closure of a set, denoted by $cl S$ or $\overline{S}$.

Th If S be any set, then $\overline{S} = S \cup S'$

Proof By defn, every pt of S is an adherent pt of S. so $S \subset \overline{S}$ — (1)

Again each limit pt of S is also an adherent pt of S and so $S' \subset \overline{S}$ .. (2)

$\therefore$ $S \cup S' \subset \overline{S}$ .... (3)

Let $p \in \overline{S}$, If $p \in S$. then $\overline{S} \subset S \cup S'$ is true.

If $p \in \overline{S}$ and $p \notin S$.

Let ~~N be a~~ $N(p)$ be a nbd of p.

Since $p \in \overline{S}$, $N \cap S \neq \phi$

$\because$ $p \notin S \Rightarrow p \notin N \cap S$

Hence ~~thus~~ there exists a nbd of p containing a point other than p. Since N is a nbd of p, it follows that each nbd of p contains a pt other than p. $\Rightarrow p \in S'$

Hence p is a limit pt of S.

$\therefore$ $\overline{S} \subset S \cup S'$

$\therefore$ $\overline{S} = S \cup S'$.

1. $\overline{N}$ = closure of $N = N \cup N' = N \cup \phi = N$

Similarly $\overline{Z} = Z$.

2. $\overline{R} = R \cup R' = R \cup R = R$.

3. $\overline{Q} = Q \cup Q' = Q \cup R = R$

4. $\overline{\phi} = \phi \cup \phi' = \phi \cup \phi = \phi$

5. $S = \{\frac{1}{n} ; n \in N\}$ $S' = \{0\}$.

$\overline{S} = S \cup S' = \{\frac{1}{n}, n \in N\} \cup \{0\}$.

Th For any ~~set~~ two sets S and T. $S \subset T \Rightarrow \overline{S} \subset \overline{T}$.

Proof Let $S \subset T$ and $p \in \overline{S}$,

$\Rightarrow p \in S \cup S' \Rightarrow p \in S$ or $p \in S'$ as $S \subset T$.

$\Rightarrow p \in T$ or $p \in T'$ $\Rightarrow S' \subset T$.

$\Rightarrow p \in S \Rightarrow p \in T$ and

$p \in S' \Rightarrow p \in T'$

$\Rightarrow p \in T \cup T' \Rightarrow p \in \overline{T}$

$\therefore$ $S \subset T \Rightarrow \overline{S} \subset \overline{T}$.

Ex Find the closure of the following sets –

i) $[a, b[$, Ans $[a, b]$

ii) $]a, b]$,, $[a, b]$

iii] $]a, \infty[$,, $[a, \infty[$

iv) $]-\infty, a[$,, $\cancel{[-\infty, a]}$. $]-\infty, a]$

2. Find the set of all adherent points of the following sets :-

i) $S = \{1 - \frac{2}{n}, n \in N\}$

$\{-1, 0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \ldots \frac{98}{100} \ldots\}$

$S' = \{1\}$

Ans $S \cup \{1\}$.

ii) $R - \{\frac{1}{n} : n \in N\}$ Ans R.

$R' = R$.

iii) $S = \{4 + (-\frac{1}{10})^n ; n \in N\}$ $S' = 4$.

$S = \{4 - \frac{1}{10}, 4 + \frac{1}{10^2}, 4 - \frac{1}{10^3}, 4 + \frac{1}{10^4} \ldots\}$

4

Ans $S \cup \{4\}$.

iv) $S = \{\frac{1}{m} + \frac{1}{n}, m, n \in N\}$ $S = \{2, \frac{3}{2}, 1, \frac{5}{6}, \frac{2}{3}, \frac{7}{12}, \frac{1}{4} \ldots\}$

$\frac{1}{2}$

$S' = \{0\}$

Ans $S \cup \{0\}$

3 Give an example of of two sets A and B

Th. III For any two sets S and T, $\overline{S \cap T} \subset \bar{S} \cap \bar{T}$. Give an example to show that $\overline{S \cap T} \neq \bar{S} \cap \bar{T}$.

Proof $S \cap T \subset S$ and $S \cap T \subset T$.

$\Rightarrow \overline{S \cap T} \subset \bar{S}$ and $\overline{S \cap T} \subset \bar{T}$.

$\Rightarrow \overline{S \cap T} \subset \bar{S} \cap \bar{T}$.

Ex $S =]a, b[$, $T =]b, c[$.

$\bar{S} = [a, b]$ $\bar{T} = [b, c]$.

$\bar{S} \cap \bar{T} = \{b\}$. $S \cap T = \{\phi\}$, $\overline{S \cap T} = \bar{\phi} = \phi$.

$\bar{S} \cap \bar{T} \cap \neq \overline{S \cap T}$.

Th The closure of a set S is a closed superset of S.

Proof $\bar{S} = S \cup S'$ $\Rightarrow$ $\bar{S}$ is a super set of S.

We are to prove that $\bar{S}$ is closed set.

~~R-S res~~ $\bar{S}$ is closed $\Leftrightarrow$ $R - \bar{S}$ is open.

$R - \bar{S}$ is the complement of $\bar{S}$ in R

Let $p \in R - \bar{S}$. $p \notin \bar{S}$.

$\Rightarrow$ for $\epsilon > 0$ $]p-\epsilon, p+\epsilon[$ contains no pts of S.

Let $q \in]p-\epsilon, p+\epsilon[$. is a nbd of q containing no pts of S, ~~q∉S~~ $q \notin \bar{S}$

q is arbitrary, no pt of $]p-\epsilon, p+\epsilon[$ can belong to $\bar{S}$.

$\therefore$ $]p-\epsilon, p+\epsilon[\subset R - \bar{S}$

$R - \bar{S}$ is a nbd of p.

p is any pt of $R - \bar{S}$, so $R - \bar{S}$ is a nbd of each of its limits.

$\therefore$ $R - \bar{S}$ is open set.

$\therefore$ $\bar{S}$ is closed.

$\therefore$ $\bar{S}$ is closed super set of S.

Th The closure of a set S is the smallest closed super-set of S.

Proof: $\bar{S}$ is closed super set of S

We are to prove $\bar{S}$ is the smallest out of all closed supersets of S.

Let T be any closed super-set of S.

We shall show $\bar{S} \subset T$.

T is closed super set of S.

$\Rightarrow$ $R-T$ is an open set

$(R-T) \cap S = \phi$

$\Rightarrow$ no pt of $R-T$ is an adherent pt of S

$\Rightarrow$ $\bar{S} \subset T$.

Th A set S is closed $\Leftrightarrow$ $\bar{S} = S$.

Proof Let S be a closed set. Then the smallest closed super set of S will be S itself $\bar{S} = S$.

Conversely let $\bar{S} = S$.

$\bar{S}$ is closed $\therefore$ S must be closed.

Th If S is a bounded set then $\bar{S}$ is a bounded.

Proof S is a bounded set.

$\Rightarrow$ there exist a real numbers h, k

s.t $A \subset [h, k]$

$\Rightarrow$ (derived set of A) $\subset$ (derived set of $[h,k]$)

$\Rightarrow$ $A' \subset [h, k]$

Now $A \subset [h, k]$ and $A \subset [h, k]$

$A \cup A' \subset [h, k]$

$\Rightarrow$ $\bar{A} \subset [h, k]$

$\Rightarrow$ $\bar{A}$ is bounded.

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Isolated Points of a set and Discrete set

If a point $p \in S$ is not a limit pt of S, then it is known as an isolated point of S. In other words $p \in S$ is an isolated pt of S if there exist a nbd of p which contains no pts of S other than p.

$S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ is a discrete set, because all its pts are isolated pts.

Each pt of any subset of R is either a limit point of S or an isolated pt. of S.

Dense (or Everywhere Dense), Dense in itself, Nowhere dense (or Non dense) and Perfect sets.

A set S is said to be dense in R if $\bar{S} = R$.

A set ~~S is ss sa~~ S is said to be dense in itself if every point of S is a limit point of S ie, if $S \subseteq S'$.

A set S is said to be nowhere dense relative to R if no nbd in R is contained in $\bar{S}$ or equivalently if the complement of $\bar{S}$ is dense in R.

ie $\overline{(\bar{S})^c} = R$.

A set S is said to be perfect if $S = S'$ or equivalently a set S is perfect when S is closed and dense in ~~itf~~ itself.

Ex 1. ~~Q~~ The set Q is dense in R

$\bar{Q} = R$; $R - Q$ is dense in R.

2. The set Q of rational numbers is dense in itself but not closed.

Similarly $R - Q$ of irrational nos dense in itself but not closed.

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3 A finite set is closed but not dense in itself.

4.]a, b[is dense in itself

5 [a, b] is closed and dense in itself and it is a perfect set.

6 If $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \ldots\}$ $n \in N$
$S' = \{0\}$ S is neither closed nor dense in itself.

7 If $S = \{0, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \ldots\}$ $n \in N$
then $S' = \{0\}$, S is nowhere dense in R because 0 is the only limit point of S and no nbd of 0 in R is contained in S.

8 The sets R and ϕ are perfect set.

9 The sets N and Z are not dense in itself.

10 The closed unbounded intervals $[a, \infty[$ and $]-\infty, a]$ are perfect sets.

11 The sets N and Z are not dense in R.

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Th For any two sets S and T

$$\overline{S \cup T} = \bar{S} \cup \bar{T}.$$

Proof $S \subset S \cup T$ and $T \subset S \cup T$

$\Rightarrow \bar{S} \subset \overline{S \cup T}$ $\quad \bar{T} \subset \overline{S \cup T}$

$\Rightarrow \bar{S} \cup \bar{T} \subset \overline{S \cup T}.$

next To prove $\overline{S \cup T} \subset \bar{S} \cup \bar{T}$ we shall prove that if a point p does not belong to $\bar{S} \cup \bar{T}$, then it cannot belong to $\overline{S \cup T}$

Let $p \notin \bar{S} \cup \bar{T}$

$\Rightarrow$ $p \notin \bar{S}$ ~~or~~ and $p \notin \bar{T}$

$\Rightarrow$ $p \notin \bar{S}$ $\Rightarrow$ there exist a nbd N_1 of p such that $N_1 \cap S = \phi$

Similarly $p \notin \bar{T}$ $\Rightarrow$, there exists a nbd N_2 of p such that $N_2 \cap S = \phi$.

Again N_1 and N_2 are nbds of p

$\Rightarrow$ $N_1 \cap N_2$ is a nbd of p.

$(N_1 \cap N_2) \cap S = \phi$

$(N_1 \cap N_2) \cap T = \phi$

$\Rightarrow (N_1 \cap N_2) \cap (S \cup T) = \phi$

$\Rightarrow$ $p \notin \overline{(S \cup T)}$

Thus $p \notin \bar{S} \cup \bar{T}$

$\Rightarrow$ $p \notin \overline{(S \cup T)}$

$\therefore$ $\overline{S \cup T} = \bar{S} \cup \bar{T}$