Isolated Points of a set and Discrete set

If a point $p \in S$ is not a limit pt of S, then it is known as an isolated point of S. In other words $p \in S$ is an isolated pt of S if there exists a nbd of p which contains no pts of S other than p.

$S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is a discrete set, because all its points are isolated pts.

Each pt of any subset of R is either a limit point of S or an isolated pt of S.

Dense A set S is said to be dense in R if $\bar{S} = R$ [$\bar{S} = S \cup S'$, S' is derived set, $\bar{S}$ closure of a set]

A set S is said to be dense in itself if every point of S is a limit point of S ie $S \subseteq S'$ (S'- derived set ie set of all limit pts).

Ex 1. A finite set is closed but not dense in itself.

2. $]a, b[$ is dense in itself

3. A set S is said to be perfect if $S = S'$ or equivalently a set S is perfect when S is closed and dense in itself.

Ex. 3. $[a, b]$ is closed and dense in itself and it is a perfect set.

4. $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \ldots\}, n \in N$, $S' = \{0\}$.
   S is neither closed nor dense in itself.

5. $S = \{0, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \ldots\}$ $n \in N$, then $S' = \{0\}$, S is nowhere dense in R, because 0 is the only limit point of S and no nbd of 0 in R is contained in $\bar{S}$.

6. The set R and ϕ are perfect set.

7. The sets N and Z are not dense in itself.

8. The closed unbounded intervals $[a, \infty[$ and $]-\infty, a]$ are perfect sets

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CANTOR NESTED INTERVAL THEOREM

For each $n \in N$, let $I_n = [a_n, b_n]$ be a non-empty closed bounded interval of real numbers such that

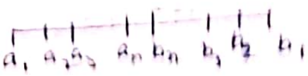
$$I_1 \supset I_2 \supset I_3 \cdots \supset I_n \supset I_{n+1} \cdots$$

and $\lim\limits_{n\to\infty}(b_n - a_n) = \lim\limits_{n\to\infty} l(I_n) = 0$

where $l(I_n)$ denotes the length of the interval I_n.

Then $\bigcap\limits_{n=1}^{\infty} I_n$ contains precisely one point.

Proof. Given $I_{n+1} \subset I_n \quad \forall\, n \in N$.



$\lim\limits_{n\to\infty}(b_n - a_n) = \lim\limits_{n\to\infty} l(I_n) = 0$

$I_n = [a_n, b_n] \qquad l(I_n) = b_n - a_n$ $\qquad b_1 \to$

$a_1 < a_2 < a_3 \cdots < a_n < \cdots$

$b_n < b_{n-1} < b_{n-2} \cdots < b_1$

Then $m > n \implies a_m < b_m \le b_n$

$m \le n \implies a_m \le a_n < b_n$

$\implies a_m < b_n \quad \forall\, m, n \in N$.

$\implies$ each b_n is upper bound for the set S

$S = \{a_1, a_2, \cdots\}$

Let $x = \sup S$

Since, each b_n is upper bound of S and

$x = \sup S$, $\quad x \le b_n \quad \forall\, n \in N$

and $a_n \le x$

$\implies a_n \le x \le b_n$ for each $n \in N$

$x \in I_n \quad \forall$ each $n \in N$

$x \in \bigcap\limits_{n=1}^{\infty} I$

We now prove that there can not be more than one point in $\bigcap\limits_{n=1}^{\infty} I_n$. If possible, let y be another point such that $y \in \bigcap\limits_{n=1}^{\infty} I_n$

Then $a_n \le y \le b_n$, $\forall\, n \in I_n$.

Let $|x - y| = \epsilon$ since $l(I_n) \to 0$ as $n \to \infty$ $\epsilon > 0$

there exist an interval $[a_m, b_m]$ of the nest such that $b_m - a_m < \epsilon$, which is impossible.

$\implies x = y$.

Note 1- If the intervals I_n are not closed, the above theorem may not hold. For example, consider the nest of open intervals $I_n =]0, 1/n[$, $n \in N$

Then $\bigcap_{n=1}^{\infty} I_n = \phi$

2. If $l(I_n)$ does not tend to zero as $n \to \infty$, then the intersection $\bigcap_{n=1}^{\infty} I_n$ may contain more than one point. For example, if ~~$I_n = -](n+1)$~~

$$I_n =]-\left(\frac{n+1}{n}\right), \frac{n+1}{n}[$$, Then $l(I_n) = 2\left(1+\frac{1}{n}\right)$

which does not tend to zero as $n \to \infty$ and hence

$\bigcap_{n=1}^{\infty} I_n = \{-1, 1\}$

$\Rightarrow$ intersection does not contain one point.