12. (a) A body moves from a fixed point O with a velocity V, under a force which produces an acceleration μx at a distance x from O. Find the time taken by the body to acquire a velocity $2V$.

[HINTS. See Art. 2.6 on page 43. The equation of motion is $\ddot{x} = \mu x$. The solution of this equation is $x = A\cosh\sqrt{\mu}t + B\sinh\sqrt{\mu}\,t$ and $\dot{x} = A\sqrt{\mu}\sinh\sqrt{\mu}\,t + B\sqrt{\mu}\cosh\sqrt{\mu}t$.

At $t = 0, x = 0, \dot{x} = V$; $\therefore$ $A = 0$ and $B = V/\sqrt{\mu}$. $\therefore$ $\dot{x} = V\cosh\sqrt{\mu}\,t$.

If T be the time when $\dot{x} = 2V$, then $2V = V\cosh\sqrt{\mu}T$, or, $T = \frac{1}{\sqrt{\mu}}\cosh^{-1}2$.]

(b) A particle moves with an acceleration which is always towards, and equal to μ divided by the distance from a fixed point O. If it starts from rest at a distance 'a' from O, show that it will arrive at O in time $a\sqrt{\pi/2\mu}$. [C.U. B.A./B.Sc.(H)'79]

[Assume that $\int_0^\infty e^{-x^2}\cdot dx = \frac{\sqrt{\pi}}{2}$.]

Answers.

3. $\pi a^{\frac{3}{2}}/6\sqrt{2\mu}$. 4. $f = \frac{u}{k^2}f^2 + \frac{4}{15k^4}f^5$; $\frac{59}{15}$ ft./sec. 7. $(a\sqrt{\mu}\sinh\sqrt{\mu}t - V\cosh\sqrt{\mu}\,t)$. $(a\cosh\sqrt{\mu}\,t - \frac{V}{\sqrt{\mu}}\sinh\sqrt{\mu}\,t)$. 8. Acceleration $= -\mu/x^2$. 12. (a) $\frac{1}{\sqrt{\mu}}\cosh^{-1}2$.

B. KINETICS

2.9. Newton's Laws of Motion

In the previous articles on *Kinematics*, the different kinds of motions concerning the geometries of motions (i.e., positions etc. of these motions) have been considered without entering into the causes which produce these motions. The motions in classical mechanics concerning the cause and effect are governed by the *three laws of Newton*. These laws were enunciated by Newton in his 'Principia Mathematica' published in the year 1686.

Newton's Laws of Motion

First Law. *Everybody continues in its state of rest or of uniform motion in a straight line, except in so far it is compelled by any external impressed force to change that state.*

Second Law. *The rate of change of momentum of a body is proportional to the impressed force, and takes place in the direction in which the force acts.*

Third Law. *To every action there is an equal and opposite reaction.*

The *first law* is also known as the '*Law of Inertia*'. The term inertia means the tendency of a body to continue as it is i.e., to remain in a state of rest or of uniform motion for ever in absence of any external force. This law gives us the qualitative definition of a force; i.e., a force is something which changes or tends to change the state of rest or of uniform motion of a body in a straight line.

The *second law* gives us a quantitative definition of a force i.e., it provides us with a measure of the applied force. The '*momentum*' of a body at any instant is defined as the product of its mass and the velocity at that instant.

To deduce the formula : P = mf from the Second Law

If P be the external force which acting on a body of mass m produces a velocity v and acceleration f at any instant t, then from the *first part* of the *second law* we have,

$$P \propto \frac{d}{dt}(mv), \quad \text{or,} \quad P = km \cdot \frac{dv}{dt},$$

where m is constant with respect to t and k is the constant of variation.

$$\text{or,} \quad P = kmf, \tag{1}$$

where $\frac{dv}{dt}$ = acceleration = f.

If the unit of force be so chosen that it acting on unit mass produces unit acceleration, i.e., $P = 1$ when $m = 1$ and $f = 1$, then from (1), $1 = k.1.1$, or, $k = 1$.

Hence from (1), we have

$$P = \text{mf}. \tag{2}$$

COROLLARY. If $P = 0$ i.e., if there is *no impressed force*, then $\frac{d}{dt}(mv) = 0$ and hence mv = constant, i.e., the body moves with constant momentum. In this case, the body moves with constant velocity.

OBSERVATIONS. Left-hand side of the equation (2) is known as the '*impressed force*' and the right-hand side as the '*effective force*'. It follows from equation (2) that if we apply forces in succession on the same mass and if they generate the same acceleration, then the forces must be equal. Again, if the same force be applied to two masses, and if it produces the same acceleration in them, then the masses must be equal. Thus, mass may be considered as the constant of proportionality between the impressed force and the produced acceleration. It also follows from equation (2) that $f = P/m$, i.e., *acceleration may be defined as the force per unit mass.*

Now from the second part of the law, we note that P and f have the same direction, i.e., an external force produces an acceleration in its direction. This is also generalised as, if two or more forces act on a body, each force being independent of other forces, produces an acceleration in its direction. This is known as *Law of Physical Independence of Forces*. Thus if a number of forces acting on a body produces equal number of accelerations in their respective directions, then equation (2) becomes

$$\Sigma\vec{P} = m\Sigma\vec{f}, \tag{3}$$

where the left hand side of equation (3) i.e., $\Sigma\vec{P}$ is the 'resultant impressed force' and the right-hand side is the 'resultant effective force'. From (3), it follows that the direction of the resultant impressed force and the direction of the resultant acceleration are the same, since $\Sigma\vec{P}$ and $\Sigma\vec{f}$ are like vectors and m is a scalar. Equation (2) is known as '*the equation of motion*'. For a particle moving along a line this equation may be written as

$$m\ddot{x} = \text{the algebraic sum of the forces along the line.}$$

By 'the algebraic sum of the forces' we mean the sum of the forces with proper signs, the sign being positive when a force is in the sense of x increasing and negative when the force is opposite to it.

The third law of motion gives us the idea that forces never exist singly, but always appear in pairs. If one body exerts a force on another, the second also exerts an equal force in the opposite direction. The first of these two forces is called the 'action' and the second one the 'reaction'. It should be noted that the action and its reaction do not act on the same body.

Units of Force

In **F.P.S.** (i.e., Foot-Pound-Second) system, the unit of force is called a **Poundal** and it is that amount of force which acting on a mass of one pound produces in it an acceleration of one foot per second per second.

In **C.G.S.** (i.e., Cm.-Gm.-Second) system, the unit of force is called a **Dyne,** and it is that amount of force which acting on a mass of 1 gm. produces in it an acceleration of 1 cm./sec^2.

$$1 \text{ poundal } = 30.48 \times 453.6 \text{ dynes.}$$

In *M.K.S.* (i.e., Metre-kg-second) system, the absolute unit of force is called a *Newton,* and it is that amount of force which acting on a mass of 1 kg. produces in it an acceleration of 1 metre/sec^2.

Weight

The **Weight** *of a body is the force with which the earth attracts the body towards its centre.*

Due to the attraction of the earth, acceleration of a freely falling body towards the earth is g. If W be the weight of a body of mass m, then by *the second law* $\mathbf{W} = \mathbf{mg}$ which always acts vertically downwards.

We shall now consider the motion in a straight line under the action of *various forces.* Let us first discuss the motion in a straight line under the action of *constant forces.*

❑ EXAMPLE 1. *A mass of* 10 *gm. falls freely from rest through* 10 *metres and is then brought to rest after penetrating* 5 *cm. of sand. Find the constant resistance of the sand in gm. weight.* [C.U. B.A./B.Sc.'77]

SOLUTION. If v cm./sec. be the velocity just before entering into sand, then

$$v^2 = 0^2 + 2g.1000 \quad [\because \ 10 \text{ metres } = 1000 \text{ cm.}]$$
$$= 2000g.$$

Let R dynes be the constant upward resistance of sand on the mass of 10 gm. when it is penetrating into the sand, and due to this resistance, let f be the retardation of the mass. Then since the mass is brought to rest after penetrating 5 cm. of sand, we have

$$0^2 = v^2 - 2f.5, \quad \text{or,} \quad 10f = v^2 = 2000\text{g}.$$
$$\therefore \quad f = 200g. \text{ cm./sec}^2.$$

The resultant force acting on the mass of 10 gm. in the upward direction is $(R-10g)$ dynes. By *Newton's Second law*, we have

$$R - 10g = 10f, \quad \text{where} \quad f = 200g \text{ cm./sec.}^2$$

$$\text{or,} \quad R = 10g + 10f = 10(g+f) = 10(g+200g) = 2010g \text{ dynes} = \mathbf{2010 \text{ gm. wt.}}$$

2.10. Pressure (or Thrust) of a body resting on a Horizontal Plane which is Moving Vertically Upwards or Downwards

[C.U. B.A./B.Sc.'80;(H)'78]

CASE I. *Let the horizontal plane be moving vertically upwards with a constant acceleration f.*

When a body of mass m is placed on the moving horizontal plane, let R be the upward reaction of the plane. As the plane is moving upwards with the constant acceleration f, the body is also moving upwards with the same constant acceleration f.

The forces acting on the body are (i) the reaction R vertically upwards and (ii) its weight mg vertically downwards. Hence he resultant force acting on the body is $(R-mg)$ vertically upwards which produces the acceleration f in the body. In this case, $R > mg$.

Hence by *Newton's second law of motion*, we have

$$R - mg = mf, \quad \text{or,} \quad \mathbf{R = m(g+f)}.$$

By *Newton's third law of motion*, the pressure of the body on the moving plane is equal and opposite to the normal reaction R of the plane and hence the pressure P exerted by the body on the plane is given by

$$\mathbf{P = m(g+f)},$$

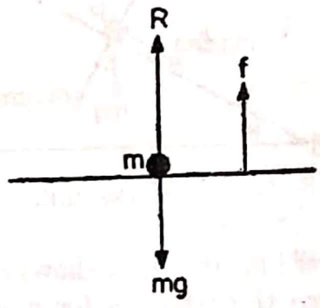


Fig. 2.10

acting vertically downwards.

CASE II. *Let the horizontal plane be moving vertically downwards with a constant acceleration f.*

In this case, the body of mass m is moving downwards with the constant acceleration f and the resultant force acting on the body is $(mg - R)$ acting vertically downwards. Clearly, $mg > R$.

Hence by *Newton's second law of motion,*

$$mg - R = mf, \quad \text{or,} \quad \mathbf{R = m(g-f)}.$$

As before, the pressure P exerted by the body on the plane is given by $\mathbf{P = m(g-f)}$, acting vertically downwards.

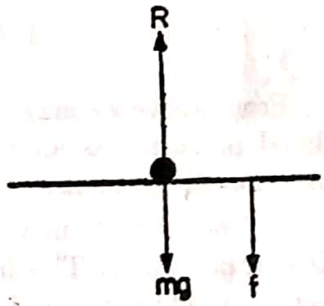


Fig. 2.11

NOTE. The reaction R is greater than or less than the weight mg of the body according as the horizontal plane is moving upwards or downwards. This explains why a man resting on an ascending lifts feels himself heavier and on a descending lift feels himself lighter than his actual weight.

If the horizontal plane be at rest or moving upwards or downwards with uniform velocity u, then $f = 0$ and $R = mg$. Hence the pressure exerted by the body on the plane is also mg.

either case. 14. 625 ft. 15. 12 ft./sec. 17. 8 secs. 18. 10[illegible] lbs. [illegible] 19. 1[illegible] miles.
23. The inclination of the radius through P is 60^0 to the vertical diameter.

2.13. Motion of Connected Systems

Let us consider the problems of moving particles connected by strings passing round pulleys. In solving these problems we define the following terms relating to strings and pulleys.

I. Light string. By the words '*light string*' we mean that the mass of the string is negligible. Though in reality, however light a string may be, it has some weight, but for all practical purposes by the word 'light string' we mean that the string is weightless. Otherwise we call it 'heavy string' and its weight should be considered.

II. Inextensible string. By the words '*inextensible string*' it is meant that the string is not stretched by applying tension i.e., Hooke's Law is not applicable in this case.

III. Light Pulley. By the words 'light pulley' it is meant that the weight of the pulley is negligible. As in the case of 'light string' there is no weightless pulley in reality. In case of 'heavy pulley' the weight of the pulley should be considered.

IV. Smooth Pulley. By the words 'smooth pully' it is meant that there is no friction between the pulley and the string over it. If the pulley is not smooth, the frictional force due to the friction on the pulley is to be considered.

For two particles connected by a 'light inextensible' string passing over a 'light smooth' pulley, it is necessary to make use of the following two facts:

I. *When two particles are hanging vertically on the two opposite sides of a pulley, their velocities and accelerations at any instant are equal in magnitude but opposite in sense.*

Let l be the length of the connecting string and at any instant x_1, x_2 are the two portions of the string on two sides of the pulley. As the string is inextensible, we have $x_1 + x_2 = l$ (Constant).

Hence, $\dot{x}_1 + \dot{x}_2 = 0$, or, $\dot{x}_1 = -\dot{x}_2$; and $\ddot{x}_1 + \ddot{x}_2 = 0$, or, $\ddot{x}_1 = -\ddot{x}_2$.

II. *The tension of a string passing over a pulley is the same throughout the string.*

Fig. 2.19

Let ρ be the mass per unit length of the string under consideration. Let us consider the motion of the portion $PQ = \Delta x$ of the string: if possible, let this portion has two unequal tensions T and $T + \Delta T$ at P and Q respectively and let it be moving with an acceleration f towards the tension $T + \Delta T$. Hence by the 2nd law of motion, we have

$$(T + \Delta T) - T = \rho \Delta x \cdot f, \quad \text{or,} \quad \frac{\Delta T}{\Delta x} = \rho f.$$

Now, in the limit when $\Delta x \to 0$ and from the consideration that the string is weightless i.e., $\rho = 0$, we have $\frac{dT}{dx} = 0$, i.e., T is constant.

NOTE. Since the pulley is smooth, therefore, on passing over the pulley there is no frictional force to alter the tension in the string on the other side of the pulley.

Next, let us consider the following simple case of a connected system.

2.14. Resulting Motion of a Connected System

Two particles of masses m_1 and $m_2 (m_1 > m_2)$ are connected by a light inextensible string passing over a light smooth fixed pulley, and are allowed to hang freely. To find the resulting motion, the tension of the string and the pressure on the pulley.

[C.U.B.Sc.'69]

Let A and B be the particles of masses m_1 and m_2 and C the fixed pulley. Since the string is light and inextensible and the pulley is smooth, the tension in the string is the same throughout its length. Let the tension be T.

Let at any instant the lengths of the strings from O, the highest point of the pulley, to A and B be x_1 and x_2 respectively. Hence the accelerations of m_1 and m_2 are respectively $\ddot{x}_1$ and $\ddot{x}_2$ downwards.

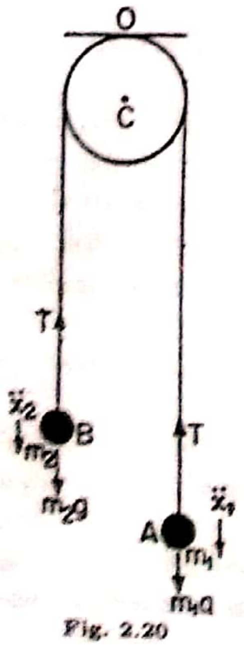


Fig. 2.20

Now, for the motion of the particle m_1, we have

$$m_1 \ddot{x}_1 = m_1 g - T. \tag{1}$$

[Since the effective force is downwards, the downward force i.e., weight is taken as positive and the upward force (i.e., tension) is taken as negative]. And, for the motion of the particle m_2, we have

$$m_2 \ddot{x}_2 = m_2 g - T. \tag{2}$$

Also, we have

$$x_1 + x_2 = \text{ the length of the string } = \text{ constant}.$$

$$\therefore \quad \dot{x}_1 + \dot{x}_2 = 0, \quad \text{and} \quad \ddot{x}_1 + \ddot{x}_2 = 0, \quad \text{or,} \quad \ddot{x}_2 = -\ddot{x}_1. \tag{3}$$

Subtracting (2) from (1), we have

$$m_1\ddot{x}_1 - m_2\ddot{x}_2 = (m_1 - m_2)g, \quad \text{or,} \quad (m_1 + m_2)\ddot{x}_1 = (m_1 - m_2)g, \quad \text{[By (3)]}$$

$$\therefore \ddot{x}_1 = \frac{\mathbf{m_1 - m_2}}{\mathbf{m_1 + m_2}}\mathbf{g}. \tag{4}$$

Since $m_1 > m_2$; $\therefore$ $\ddot{x}_1 > 0$ and hence the particle m_1 will move downwards with an acceleration given by (4).

Also from (3), $\ddot{x}_2 = -\ddot{x}_1$; hence the particle m_2 will move upwards with the same acceleration as m_1.

Again, from (1)

$$T = m_1 g - m_1\ddot{x}_1 = m_1 g - m_1 \cdot \frac{m_1 - m_2}{m_1 + m_2}.g = \frac{2m_1 m_2}{m_1 + m_2}g, \quad \text{[from (4)]} \tag{5}$$

i.e., $\mathbf{T} = \dfrac{\mathbf{2m_1m_2}}{\mathbf{m_1 + m_2}} \cdot g$ **which gives the tension in the string.**

Since the string pulls the pulley downwards on both sides by a force equal to the tension in the string; hence **pressure on the pulley** = the resultant of two equal like parallel forces, each being T is

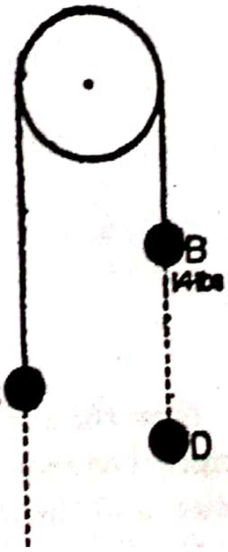
$$\mathbf{2T} = \frac{\mathbf{4m_1m_2}}{\mathbf{m_1 + m_2}}\ \mathbf{g}. \tag{6}$$

NOTE 1. It follows from (4), (5) and (6) that the acceleration of the masses, tension in the string and pressure on the pulley are all independent of time i.e., they remain constant throughout the motion.

NOTE 2. If P be the pressure on the pulley and W be the sum of the weights of the particles, then $W - P = (m_1 + m_2)g - \frac{4m_1m_2}{m_1+m_2}g = \frac{(m_1-m_2)^2}{m_1+m_2}g > 0$ unless $m_1 = m_2$. Hence, $W > P$ when $m_1 \neq m_2$; i.e., for an accelerated system the pressure on the pulley is less than the total weight of the particles.

It is to be noted that the relation between the displacements of the particles and the length of the string must be established to discuss the motion of connected systems.

❑ EXAMPLE 1. *A string having at its ends two particles of masses 14 lbs. and 7 lbs. passes over a smooth pulley. If the string breaks after the motion has continued for 3 secs., find after what further interval of time the smaller mass comes to its original position.*



SOLUTION. Before the string breaks the acceleration f of the masses is

$$f = \frac{14 - 7}{14 + 7} \times 32 = \frac{32}{3}\ \text{ft./sec}^2.$$

Let, at the beginning the masses 7 lbs. and 14 lbs. were at A and B and after 3 secs. they are at C and D respectively.

When the string breaks, the smaller mass at C has an upward velocity $u + ft = \frac{32}{3} \times 3 = 32$ ft./sec.

[$\because$ $u = 0,\ t = 3$] and

$$AC = ut + \frac{1}{2}ft^2 = \frac{1}{2} \times \frac{32}{3} \times 3^2 = 48 \text{ ft.}$$

ANALYTICAL DYNAMICS

CHAPTER I

WORK, POWER AND ENERGY

1·1. Work.

Work done by a force acting at a point of a body for any time is the product of the force, and the displacement of the point of application of the force during that time in its own direction.

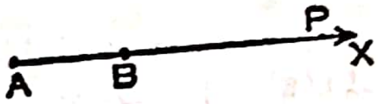


Fig. (i)

Fig. (ii)

Let a force P be acting on a body at A in the direction AX for any time, and let A move to B during the interval. If AB be in the direction AX, as in the first figure, the work done $= P.AB$, and is *positive*. If the displacement AB of A is in a direction opposite to the direction of P, as in the second figure, the displacement measured in the direction of P is $-AB$, and the work done by the force here is $-P.AB$, which is *negative*.

If the displacement AB be in a direction different from the direction of the force, say, making an angle θ with AX as in the third figure, the displacement measured in the direction of P is $AN = AB \cos\theta$, and in this case we get more generally,

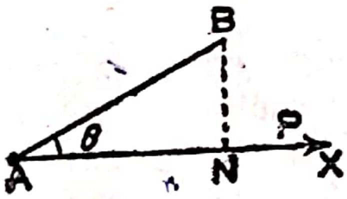


Fig. (iii)

Work done by $P = P.AB \cos\theta = AB.P \cos\theta$

= *Force* × *component of displacement of its point of application along the line of action of the force*

= *Total displacement* × *component of the acting force along the direction of displacement.*

Note. Evidently the work done is positive if θ be acute, and negative if θ be obtuse. In particular, if $\theta=90°$, the work done is zero, *i.e.*, no work is done by a force if the resultant displacement of its point of application is perpendicular to the line of action of the force.

If the displacement or its component is in a direction opposite to that of the acting force, work is said to be *done against* the force.

Analytically :

If the particle moves along a straight line which we may take as the x-axis from the point x_1 to the point x_2 in time t, expression for the work done in time t can be written analytically as

$$\int_{x_1}^{x_2} F\, dx,$$

where F is the component (constant or variable) of the force acting upon the particle along the x-axis in any position.

In the most general case, if a particle describes a smooth curve under any force and if F be the component of the force along the tangent to the path at any instant and ds be the element of the length along the path described in an infinitesimal time dt, then during any interval of time from t_1 to t_2, the corresponding space described being s_1 to s_2, the total work done is

$$\int_{s_1}^{s_2} F\, ds=\int_{t_1}^{t_2} F.v\, dt$$

whether F be constant or variable.

In case of the motion of a particle along a curve if (x, y) be co-ordinates of the position of the particle referred to rectangular axes at any instant and X, Y be the components (constant or variable) of the resultant force acting upon the particle at that instant, then as the particle moves from the point (x_1, y_1) to the point (x_2, y_2) on its path, the total work done is

$$\int_{(x_1, y_1)}^{(x_2, y_2)} (X\, dx+Y\, dy),$$

where X and Y are not constants, they are usually known functions of (x, y).

1·2. Units for measurement of Work.

When a force of one poundal acting on a body displaces its point of application through one foot in its own direction, the amount of work done is defined to be a **Foot-poundal.** This is the British absolute unit of work.

When a force equal to the weight of one pound displaces the point of application through one foot in its own direction, the work done is defined to be one **Foot-pound.** For instance, when a man raises a mass of one pound vertically upwards through one foot, he does work of one foot-pound against the force of gravity, whereas the work done by the weight of the body in this case is negative, and = − 1 ft.-lb.

As 1 lb. wt. = g poundals, it is clear that

1 ft.-lb. = g foot-poundals.

When a force of one dyne acting on a body displaces its point of application through one centimetre in its own direction, the amount of work done is called an **erg.** This is the c.g.s. absolute unit of work.

As this is very small, a bigger unit of c.g.s. system is

one joule = 10^7 ergs.

As one poundal = 13800 dynes roughly,

1 foot-poundal = 30·48 × 13800 ergs

= 420624 ergs approximately,

and 1 ft.-lb. = $\frac{32 \times 420624}{10^7}$, *i.e.*, 1·346 Joules nearly.

1·3. Power.

When an agent (say, a man, or a machine, or an engine) *is doing work continuously, the rate at which it does work per unit of time is defined to be its power.*

BRITISH UNIT—*When an agent is doing work at the rate of 550 foot-pounds per second, it is said to have one* **Horse-power** (briefly 1 H.P.).

C.G.S. UNIT—*When an agent does work at the rate of 1 Joule* (10^7 *ergs*) *per second, its power is said to be one* **Watt.**

We can show easily that 1 H.P. = 746 Watts neraly.

From definition, it follows that

$$Power = Force \times Velocity.$$

1·4. Energy.

Energy *of a body is its capacity for doing work.*

There are two kinds of energy that a body may possess, namely, Kinetic and Potential.

A moving body, by virtue of its motion, possesses a certain capacity for doing work. For, if a force be applied to stop it, it does not stop immediately, but moves a certain distance against the force before it stops. Consequently it does a certain amount of work against the force before coming to rest, and hence at the iniitial moving state it had in it a capacity for doing this amount of work, *i.e.*, it possessed an energy. If the opposing force be greater or less, the distance moved by the body before coming to rest will be less or greater, and it will be seen below that the amount of the work which that body will perform is definite.

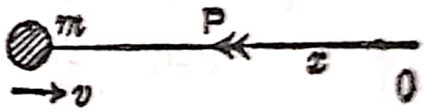
Again, for a body acted on by a given system of forces we may contemplate a suitable position as the standard position. If the body be displaced from this position to some other position, in general a certain amount of work will have to be done against the acting forces. If the body be allowed to go back to the former standard position, the acting forces will do in their turn the above amount of work. The capacity for doing this amount of work then was stored up in the body in its displaced position, which becomes manifest as the body is allowed to go back to its standard position. Thus, a body may possess energy due to its position. We then formally define the two kinds of energy as follows :

Kinetic Energy *is the capacity for doing work, which a moving body possesses by virtue of its motion, and is measured by the work which the body can do against any force applied to stop it, before its velocity is destroyed.*

Potential Energy *of a body is the capacity for doing work, which it possesses by virtue of its position or configuration, and is measured by the amount of work which the system of forces acting on the body can do in bringing the body from its present position to some standard position.*

1·5. *The kinetic energy of a body of mass m moving with a velocity v is* $\frac{1}{2}mv^2$ *(in absolute units).*

Imagine a force P to be applied against the direction of motion of the body of mass m moving with a velocity v. Let x be the distance advanced by the body before its velocity is destroyed. Then, since the opposing acceleration produced by the force is P/m, we have



$$0=v^2-2\,(P/m)\,x, \text{ whence, } Px=\tfrac{1}{2}mv^2.$$

Thus, the work done by the body against the force before it comes to rest is $\frac{1}{2}mv^2$, and this is, by definition, the measure of the kinetic energy of the body.

It may be noted that the K.E. ultimately depends on m and v, but not on P.

Note 1. It is seen from above that the *unit of energy* is the same as that of work in absolute units (for which $P=mf$ holds) and is therefore usually in *foot-poundals* or ergs.

Note 2. The term Vis Viva is used to denote *twice the Kinetic Energy* of a body, so that Vis Viva $=mv^2$.

1·6. The Principle of Energy.

The change in the kinetic energy of a body is equa[l to] the work done by the acting force.

Let a force P act on a body of mass m for any ti[me] and let u be the initial velocity and v the velocity at [the] end of the interval, along the line of action of the fo[rce]. Let x be the displacement of the body in that direc[tion] during the interval. The acceleration produced is $\frac{P}{m}$ and so

$$v^2 = u^2 + 2\frac{P}{m} \cdot x.$$

Hence, $\frac{1}{2}\mathbf{mv}^2 - \frac{1}{2}\mathbf{mu}^2 = \mathbf{Px}$.

Now $\frac{1}{2}mv^2$ and $\frac{1}{2}mu^2$ are respectively the final and initi[al] kinetic energy of the body and Px represents the work do[ne] by the acting force. Hence, the required result is proved.

Analytically, from the equation of motion

$$P = mv\frac{dv}{dx},$$

integrating w. r. to x, between the limits x_1 to x_2, if v_1 an[d] v_2 be the velocities at those points.

$$\int_{v_1}^{v_2} mv\,dv = \int_{x_1}^{x_2} P\,dx$$

i.e., $\frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2 = P(x_2 - x_1)$,

or, $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = Px$, as above.

Note. The above result which is sometimes spoken of as "*Ener[gy] equation*" may also be put in the form

$$\frac{\frac{1}{2}mv^2 - \frac{1}{2}mu^2}{x} = \text{P},$$

which may be expressed as follows :

The change in kinetic energy per unit space is equal to the acti[ng] force.

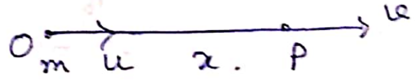
The Principle of Energy

The change ~~of~~ in K.E of a body is equal to the work done by the acting force.

Case 1 :- When a particle moves along a straight-line in the direction of the acting force

mass = m.

force = F (constant)



x = displacement ~~during~~ in an interval of time t

u = initial vel.

v = velocity at the end of the interval.

acceleration $= \frac{F}{m}$.

$v^2 = u^2 + \frac{2F}{m} x$.

$\therefore \frac{1}{2} m v^2 - \frac{1}{2} m u^2 = Fx \quad \text{--} (1)$.

Fx = work done by the force F acting on the particle m.

$\frac{1}{2} m v^2$ = final K.E , $\frac{1}{2} m u^2$ = initial K.E .

Eqn (1) can also be derived on integrating the eqn of motion

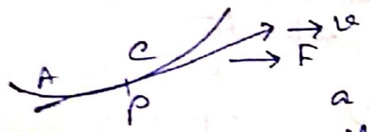
$F = m v \frac{dv}{dx}$

$\Rightarrow m v \, dv = F \, dx$

$\Rightarrow m \int_u^v v \, dv = \int_0^x F \, dx$

$\Rightarrow \frac{1}{2} m (v^2 - u^2) = Fx$.

Case 2 When a particle moves along a plane curve



A particle of mass m moves along a curve C. At any time t, let P be the position of the particle and v be the velocity along the tangent at P.

Let $\vec{r_0}$ and $\vec{u}$ be the values of the position vector $\vec{r}$ and velocity $\vec{v}$ of the particle at time $t = 0$.

$\vec{F}$ = the resultant force acting at the point on the curve C whose position vector is $\vec{r}$

work done $= \int_{r_0}^{r} \vec{F} \cdot dr$

eqn of motion $m \frac{d\vec{v}}{dt} = \vec{F}$

work done =

Simple Problems. :-

$$\frac{d\vec{v}}{dt}\cdot d\vec{r} = \frac{d\vec{v}}{dt}\cdot\frac{d\vec{r}}{dt}\cdot dt$$

$$= \vec{v}\,\frac{d\vec{v}}{dt}\cdot dt \quad \left[\vec{v} = \frac{d\vec{r}}{dt}\right]$$

$$= \frac{d}{dt}\left(\frac{\vec{v}^2}{2}\right)dt.$$

$$\text{work done} = \int_{r_0}^{r} m\, d\left(\frac{v^2}{2}\right) = \frac{1}{2}\left[v^2\right]_u^v$$

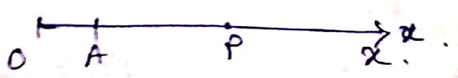
$$= \frac{1}{2}m\left(v^2 - u^2\right).$$

Conservative Forces and the Principle of Conservation of Energy

A system of forces acting on a body is defined to be conservative when work done by the forces of the system while the body moves from ~~f~~ one position to another depends on these two positions only but is independent of the path along which the body moves.

When the body moves along a closed path, the work done by the forces acting on the body is zero.

Case 1 :- For a particle of mass m moving along a st. line under a constant force F, the sum of the K.E and P.E of the particle is constant at any point of its path.



f is the accel^n produced by F

m = mass.

~~u & v are initial & final velocity~~

u = initial vel. at A

v = velocity at P.

OA = a

OP = x

$$K.E = \frac{1}{2}mv^2 = \frac{1}{2}m\left(u^2 + 2fx\right)$$

$$P.E = F(a - x)$$

$$K.E + P.E = \frac{1}{2}m\left(u^2 + 2fx\right) + mf(a - x)$$

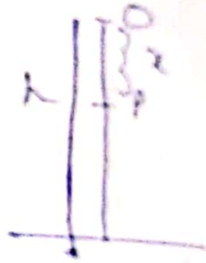
$$= \frac{1}{2}mu^2 + mfa.$$

At O, $K.E = \frac{1}{2}mu^2$

$P.E = mfa.$

Hence the proof

Case 2 For a particle of mass m falling from rest under gravity from a height h above the ground.



v = vel. at P.

$v^2 = 2gx$

$K.E = \frac{1}{2}mv^2 = \frac{1}{2}m\,2gx = mgx$.

$P.E = mg(h-x) =$

At P → $K.E + P.E = mgx + mg(h-x) = mgh$.

∵ initial vel. 0,

∴ at O K.E = 0.

P.E = mgh.

At O → $K.E + P.E = 0 + mgh = mgh$.

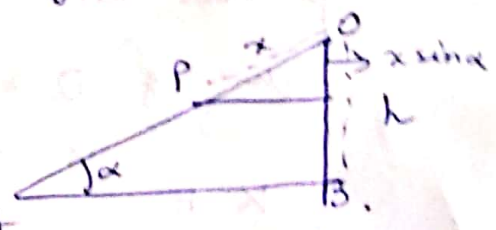
Just before the impact on the ground [x = h)

K.E = mgh

P.E = mg.0

∴ K.E + P.E = mgh. Hence the proof

Case 3 For a particle of mass m moving down a smooth inclined plane under gravity alone



The particle starts with a vel u at a pt O and its vel be v when it has moved a distance OP = x down the plane.

h = vertical height of O above the ground

α = inclination of the plane to the horizon.

$K.E = \frac{1}{2}mv^2$

$P.E = mg(h - x\sin\alpha)$

$v^2 = u^2 + 2gx\sin\alpha$.

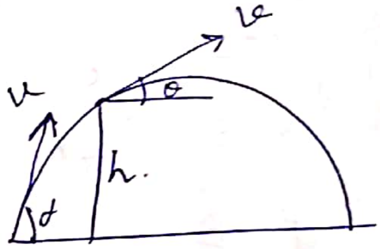
∴ $K.E = \frac{1}{2}m(u^2 + 2gx\sin\alpha)$

$K.E + P.E = \frac{1}{2}mu^2 + mgh$.

= K.E at O + P.E at O.

Case 4 :- ~~For a particle of mass m projected in ... from the Horizon~~ "

For a projectile :-



Let a particle of mass m be projected from the ground with a velocity u at angle α to the horizon

$$\text{Initial K.E} = \frac{1}{2}mu^2$$
$$\text{P.E} = 0.$$
$$\text{K.E} + \text{P.E} = \frac{1}{2}mu^2.$$

v be the velocity of the projectile at an angle θ with the horizon, when it is at any vertical height h above the ground

∵ There is no horizontal accelⁿ of the projectile, its horizontal component of velocity remains unchanged

$$\therefore \quad v\cos\theta = u\cos\alpha \quad \text{..(i)}$$

The accelⁿ due to gravity (g) being downwards, the motion of the projectile in the vertically upwards direction :-

$$v^2\sin^2\theta = u^2\sin^2\alpha - 2gh. \quad \text{..(ii)}$$

squaring (i) & adding to (ii)

$$v^2 = u^2 - 2gh.$$

$$\text{K.E} = \frac{1}{2}mv^2 = \frac{m}{2}(u^2 - 2gh)$$

h = vertical height above the ground

$$\text{P.E} = mgh$$

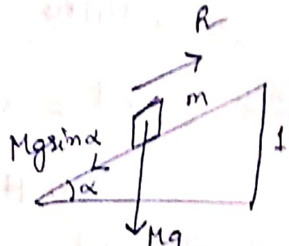
$$\therefore \text{K.E} + \text{P.E} = \frac{1}{2}mu^2.$$

= the initial total energy of the projectile, and the same at all heights.

Simple Problems :-

Ex A cyclist and his machine together are of mass M lb. If he rides without pedalling down an incline of 1 in m with a uniform speed of v ft/sec, show that to go up an incline of 1 in n at the same rate he must work at a rate equal to

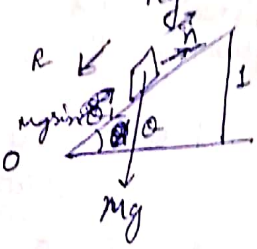
$$M\left(\frac{1}{m}+\frac{1}{n}\right)\frac{v}{550}\ \text{H.P.}$$



Cyclist rides without pedalling down the inclined plane with a uniform speed v,

$$P = mf = 0 \quad \because f=0.$$

$$\sin\alpha = \frac{1}{m}$$



R be the resistance, $Mg\sin\alpha - R = 0$

$$Mg\sin\alpha = R.$$

$$R = \frac{Mg}{m}\ \text{Poundals.}$$

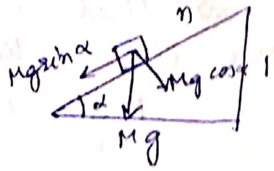
The cyclist is ascending

$$F = mg\sin\theta + R = \frac{Mg}{n}+\frac{Mg}{m} = Mg\left(\frac{1}{m}+\frac{1}{n}\right)$$

work done $F.v = Mgv\left(\frac{1}{m}+\frac{1}{n}\right) = Mv\left(\frac{1}{m}+\frac{1}{n}\right)$ ft lbs.

$$= \frac{Mv}{550}\left(\frac{1}{m}+\frac{1}{n}\right)\ \text{H.P.}$$

Ex A train of mass M lbs is ascending a smooth incline of 1 in n when the velocity of the train is v ft/sec, its acceleration is f ft/sec². Prove that the effective horse power of the engine is $Mv(nf+g)/550gn$.



$$\sin\alpha = \frac{1}{n}.$$

vel. of the train = v ft/sec

accel " " " f ft/sec².

P = force exerted by the engine

Resultant force $P - Mg\sin\alpha = Mf.$

$$P = Mg\sin\alpha + Mf = Mg.\frac{1}{n} + Mf.$$

work done $= P.v = \frac{Mv}{n}(nf+g)$ ft. Poundal

let H be the req. Horse Power,

work done $H \times 550 \times g = \frac{Mv}{n}(nf+g)$

$$H = \frac{Mv(f+g)}{550gn}.$$

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Ex If the mass of a train is M tons, the engine works at a Horse Power H, and the resistance is $(a+bv^2)$ lb.wt. per ton, where v is the velocity in miles/hr. and a and b are constants, prove that the the acceleration when the train moves at v miles/hr is

$$\left(\frac{75H}{14Mv} - \frac{a+bv^2}{70}\right) ft/sec^2$$

Ans vel $= v$ m/hr $= \frac{v \times 1760 \times 3}{60 \times 60}$ ft/sec $= \frac{22v}{15}$ ft/sec

$550H = P \cdot \frac{22v}{15}$

Force exerted by the engine P $= \frac{15 \times 550 H}{22v} = \frac{15 \times 25 H}{v}$

f be acceleration

$2240 \times M \times f = \frac{15 \times 25}{v} Hg - Mg(a+bv^2)g$

$= \left[\frac{375 H}{v} - M(a+bv^2)\right] g$

$\therefore f = \frac{1}{2240 M}\left[\frac{375 H}{v} - M(a+bv^2)\right] g$ ft/sec²

~~$= \frac{1}{2240}\left(\frac{375 H}{Mv} - \frac{a+bv^2}{}\right) \times 32$ ft/sec²~~

$= \left(\frac{375 H}{70 Mv} - \frac{a+bv^2}{70}\right)$ ft/sec²

$= \left(\frac{75 H}{14 Mv} - \frac{a+bv^2}{70}\right)$ ft/sec²

Prob 1 An engine working at a constant rate H draws a load of mass M against a resistance R. Show that maximum speed attained is $\frac{H}{R}$ and the time taken to attain half this speed is $\frac{MH}{R^2}\left[\log 2 - \frac{1}{2}\right]$.

Ans v be the vel at time t

P " " force exerted by it

$Pv = H$

Resultant force $P - R = \frac{H}{v} - R$

eqⁿ of motion $M\frac{dv}{dt} = \frac{H}{v} - R$

v is max^m –

$$\frac{dv}{dt} = 0$$

$$\therefore \frac{H}{v} - R = 0$$

$$v_{max} = \frac{H}{R}.$$

$$\frac{M\,dv}{dt} = \frac{H}{v} - R$$

$$dt = \frac{Mv}{H - vR}\,dv$$

Int

$$\int_0^{t_0} dt = \int_0^{v=\frac{H}{2R}} \frac{Mv}{H - vR}\,dv$$

$$\therefore t_0 = -\frac{M}{R}\int_0^{\frac{H}{2R}} \frac{H - vR - H}{H - vR}\,dv$$

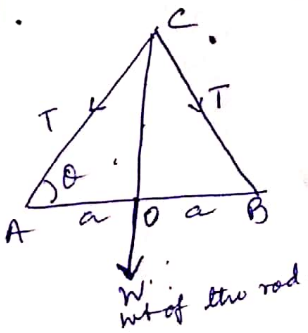
$$= \frac{M}{R}\int_0^{\frac{H}{2R}} \frac{H\,dv}{H - vR} - \frac{M}{R}\int_0^{\frac{H}{2R}} dv$$

$$= \left[\frac{-MH}{R^2}\log(H - vR)\right]_0^{\frac{H}{2R}} - \frac{M}{R}\cdot\frac{H}{2R}$$

$$= \frac{M}{R}\left[\frac{H}{R}\log\frac{H}{H/2} - \frac{H}{2R}\right]$$

$$= \frac{MH}{R^2}\left[\log 2 - \frac{1}{2}\right].$$

Ex If an elastic string, whose natural length is that of a uniform rod, be attached to the rod at both ends and suspended by the middle point, show by means of the Principle of Energy, that the rod will sink until the strings are inclined to the horizon at an angle θ given by the eq^n $\cot^3\theta/2 - \cot\theta/2 = 2n$, given that the modulus of elasticity of the string is n times the weight of the rod.



AB = rod = $2a$
C = middle pt of the string

$OC = a\tan\theta$

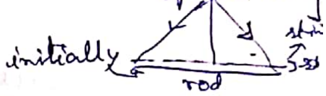
work done = $W.OC = Wa\tan\theta$

work done against the tension of the string :-

$$= 2 \times \left(\text{mean of the initial \& final tension}\right) \times \text{extension produced.}$$

$$= 2 \times \frac{1}{2}\left[0 + \frac{\lambda(AC - AO)}{AC}\right](AC - AO)$$

[length of the string = length of the rod]



$$= \frac{\lambda (AC - AO)^2}{AC}$$

$$= \lambda \cdot \frac{(a\sec\theta - a)^2}{a} \qquad \lambda = nW$$

$$= n.aW(\sec\theta - 1)^2$$

Work done by the rot of the rod W = the word done against the tension

$$aW\tan\theta = anW(\sec\theta - 1)^2$$

$$\Rightarrow \quad = anW\frac{(1-\cos\theta)^2}{\cos^2\theta}$$

$$\Rightarrow \sin\theta\cos\theta = n\, 4.\sin^4\theta/2$$

$$\Rightarrow 2\sin\theta/_2\cos\theta/_2\left(\cos^2\theta/_2 - \sin^2\theta/_2\right) = n4\sin^4\theta/_2$$

$$\Rightarrow \cot^3\theta/_2 - \cot\theta/_2 = 2n$$

Ex An engine is pulling a train and works at a constant power doing H units of work per second. If M be the mass of the whole train and F the resistance supposed to be constant, show that the time of generating the velocity v from rest is

$$\left(\frac{MH}{F^2}\log\frac{H}{H-Fv} - \frac{Mu}{F}\right) \text{ seconds.}$$

Let P = force exerted by the engine

Resultant force $P - F = \frac{H}{v} - F$.

Eq^n of Motion $M\frac{du}{dt} = \frac{H}{v} - F$

$$\int_0^t dt = \int_0^u \frac{Mu}{H - Fv}\,du \qquad \begin{matrix} t=0 & t=t \\ v=0 & u=u \end{matrix}$$

$$t = -\frac{M}{F}\int_0^u \frac{H - Fu - H}{H - Fv}\,du = \frac{MH}{F}\int_0^u \frac{du}{H-Fv} - \frac{M}{F}\int_0^u du$$

$$= -\frac{MH}{F^2}\Big[\log(H - Fv)\Big]_0^u - \frac{M}{F}u$$

$$= \frac{MH}{F^2}\left[\log\frac{H}{H-Fv}\right] - \frac{Mu}{F} \text{ seconds.}$$