Characteristic Equation:

Matric Polynomial :- An expression of the form
$$F(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_{m-1} x^{m-1} + A_m x^m$$

~~where A_0, $A_1x + A_2x^2 + \cdots + A_{m-1}x^{m-1} + A_m x^m$,~~

where $A_0, A_1, A_2, \cdots, A_m$ are all square matrices of the same order is called a matric polynomial of degree m, provided A_m is not a null matrix. For example

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x^2 + \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} x^3$$

is a matric polynomial of degree 3.

Characteristic Equation :- Let A be a square matrix of order n over the field F, then the polynomial $A - \lambda I$, where I is the unit matrix of order n, is called the characteristic matrix of A, λ being a scalar.

The determinant $|A - \lambda I|$ which is an ordinary polynomial in λ of nth degree, with scalar coefficients, is called the characteristic polynomial of A, the equation

$|A - \lambda I| = 0$ is called the characteristic equation of the matrix A. The degree of the characteristic equation is the same as the order of the matrix A.

The roots of the characteristic equation are called the eigen values or characteristic roots of the matrix A.

[**Caley-Hamilton Theorem :** Every square matrix satisfies its own characteristic equation.] Important.

This theorem states that, if

$$|A - \lambda I| = (-1)^n (\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \cdots + p_n) = 0 \quad -(1)$$

be the characteristic equation, then

$$A^n + p_1 A^{n-1} + p_2 A^{n-2} + \cdots + p_{n-1} A + p_n = \underline{0} \quad --(2)$$

Eigen vector :- Let A be an $n \times n$ matrix over a field F.

Let X be a vector which transforms to its multiple λX, by means of the square matrix A. ie

$$\lambda X = AX \quad \text{where } \lambda \text{ is a scalar.}$$

Then $$AX - \lambda I X = 0$$

or $$(A - \lambda I) X = 0. \quad \text{--- (1)}$$

Obviously, for any value of λ, the null vector $X = 0$ is a solution of (1). A value of λ for which (1) has a solution $X \neq 0$ is called an eigen value or characteristic value of the matrix A. The corresponding solution $X \neq 0$ of (1) are called eigen vectors or characteristic vectors of A corresponding to the eigen value λ.

Determination of scalar λ and nonzero vector X, satisfying the equation $AX = \lambda X$ is known as the eigen value problem.

Ex Verify Caley-Hamilton theorem for the matrix A. Express A^{-1} as a polynomial in A and then compute A^{-1}.

i) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$ ii) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} \rightarrow$ HW

Characteristic eqn $$|A - \lambda I| = 0$$

$\lambda I = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

or $$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ 2 & 3 & 2-\lambda \end{vmatrix} = 0$$

or $\cancel{1.[(2-\lambda)(2-\lambda) - 3] + 0 + 0 = 0}$

$\cancel{\lambda^2 - 4\lambda + 1 = 0}$. which is the characteristic equation.

Now, we are to show the matrix A satisfies the eq^n ①.

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{pmatrix}$$

$$\therefore \cancel{A^2 - 4A + 4I = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 7 & 7 \\ 9 & 12 & 12 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\cancel{= \begin{pmatrix} 1-4+4 & 0 & 0 \\ 5-4+0 & & \\ & & \end{pmatrix}}$$

$$(1-\lambda)\{(2-\lambda)(2-\lambda)-3\} = 0$$

$$\Rightarrow (1-\lambda)(4+\lambda^2-4\lambda-3) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-4\lambda+1) = 0$$

$$\Rightarrow -\lambda^3 + 4\lambda^2 - \lambda + \lambda^2 - 4\lambda + 1 = 0$$

$$\Rightarrow -\lambda^3 + 5\lambda^2 - 5\lambda + 1 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 5\lambda - 1 = 0. \quad \text{--} ①$$

which is the characteristic equation.

$$\therefore A^3 = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 20 & 26 & 15 \\ 35 & 45 & 26 \end{pmatrix}$$

$$\therefore A^3 - 5A^2 + 5A - I$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 20 & 26 & 15 \\ 35 & 45 & 26 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-5+5-1 & 0-0+0-0 & 0-0+0-0 \\ 20-25+5+0 & 26-35+10-1 & 15-20+5+0 \\ 35-45+10+0 & 45-60+15+0 & 26-35+10-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$\therefore$ matrix A satisfies the characteristic eqn

Now

$$A^3 - 5A^2 + 5A - I = 0$$

$$\therefore \; A^{-1}(A^3 - 5A^2 + 5A - I) = 0$$

$$\therefore \; A^2 - 5A + 5I - A^{-1} = 0$$

$$\therefore, \; A^{-1} = A^2 - 5A + 5I$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 5 & 7 & 4 \\ 9 & 12 & 7 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-5+5 & 0-0+0 & 0-0+0 \\ 5-5+0 & 7-10+5 & 4-5+0 \\ 9-10+0 & 12-15+0 & 7-10+5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$$

AA^{-1}

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I.$$

$$\therefore \; A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}.$$

HW Find the characteristic equations of the matrices

i) $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Show that the equations are satisfied by A and hence obtain the inverse of the given matrices.

Submit on 7/4/2020 at 6-30 p.m