

Mathematical methods for economic theory

Martin J. Osborne



CONTENTS

TEXT

EXERCISES



3.4 Quasiconcavity and quasiconvexity

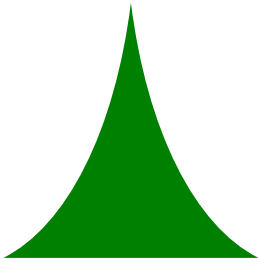
Definitions and basic properties

Think of a mountain in the Swiss Alps: cows grazing on the verdant lower slopes, snow capping the majestic peak.

Now forget about the cows and the snow. Ask yourself whether the function defining the surface of the mountain is concave. It is if every straight line connecting two points on the surface lies everywhere on or under the surface.

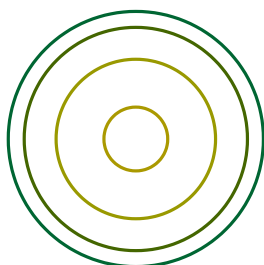
If, for example, the mountain is a perfect dome (half of a sphere), then this condition is satisfied, so that the function defined by its surface is concave. The condition is satisfied also if the mountain is a perfect cone. In this case, every straight line connecting the peak to another point on the surface lies exactly *on* the surface.

Now suppose that the mountain is a deformation of a cone that gets progressively steeper at higher altitudes—call it a “pinched cone”. (Many mountains seem to have this characteristic when you try to climb them.) That is, suppose that when viewed from far away, the mountain looks like this:

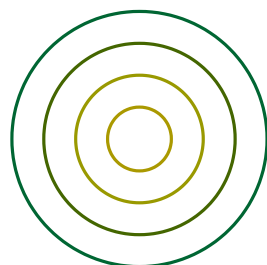


In this case, a straight line from the top of the mountain to any other point on the surface does *not* lie on or under the surface, but rather passes through clear air. Thus the function defined by the surface of the mountain is *not* concave.

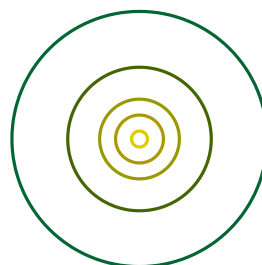
The function does, however, share a property with a perfect dome and a cone: on a topographic map of the mountain, the set of points inside each contour—the set of points at which the height of the mountain exceeds any given number—is convex. In fact, each contour line of this mountain, like each contour line of a perfect dome and of a cone, is a circle. With contour lines drawn for regularly-spaced heights (e.g. 50m, 100m, 150m, ...), topographic maps of a dome, cone, and pinched cone are given in the following figure. If you want to see a 3D representation of any of the objects, google the formula given under the corresponding figure.



Dome
For 3D, google
 $z = -(x^2 + y^2)$



Cone
For 3D, google
 $z = -(x^2 + y^2)^{0.5}$



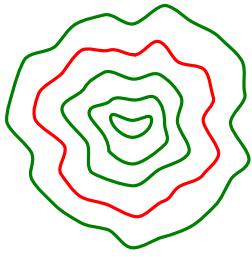
Pinched cone
For 3D, google
 $z = -(x^2 + y^2)^{0.2}$

The spacing of the contour lines differs, but the set of points inside every contour has the same shape for each mountain—it is a disk. In particular, every such set is convex.

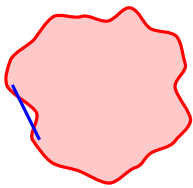
If we model the surface of the mountain as a function f of its longitude and latitude (x, y) , then a contour is a level curve of f . A function with the property that for every value of a the set of points (x, y) such that

$f(x, y) \geq a$ —the set of points inside every contour on a topographic map—is convex is said to be *quasiconcave*.

Not every mountain has this property. In fact, if you take a look at a few maps, you'll see that almost *no* mountain does. A topographic map of an actual mountain is likely to look something like this:



The contours of this mountain do not enclose convex sets. Take, for example, the one in red. The blue line, connecting two points in the set enclosed by the contour, goes outside the set.



Thus the function defined by the surface of this mountain is not quasiconcave.

Let f be a multivariate function defined on the set S . We say that f (like the function defining the surface of a mountain) is *quasiconcave* if, for any number a , the set of points for which $f(x) \geq a$ is convex.

Definition

Let f be a function of many variables defined on the set S . For any real number a , the set

$$P_a = \{x \in S: f(x) \geq a\}$$

is called the **upper level set** of f for a .

In the case of a mountain, P_a is the set of all points at which the altitude is at least a .

Example

Let $f(x, y) = -x^2 - y^2$. The upper level set of f for a is the set of pairs (x, y) such that $-x^2 - y^2 \geq a$, or $x^2 + y^2 \leq -a$. Thus for $a > 0$ the upper level set for a is empty, and for $a < 0$ it is a disk of radius $(-a)^{1/2}$.

Definition

The function f of many variables defined on a convex set S is **quasiconcave** if **every upper level set** of f is convex. (That is, $P_a = \{x \in S: f(x) \geq a\}$ is convex for every value of a .)

The notion of *quasiconvexity* is defined analogously. The lower level set of f for a is the set of all the points that yield a value for the function of at most a .

Definition

Let f be a function of many variables defined on the set S . For any real number a , the set

$$P^a = \{x \in S: f(x) \leq a\}$$

is called the **lower level set** of f for a .

In the case of a mountain, P^a is the set of all points at which the altitude is at most a .

Definition

The function f of many variables defined on a convex set S is **quasiconvex** if **every lower level set** of f is convex. (That is, $P^a = \{x \in S: f(x) \leq a\}$ is convex for every value of a .)

Note that f is quasiconvex if and only if $-f$ is quasiconcave.

The notion of quasiconcavity is weaker than the notion of concavity, in the sense that every concave function is quasiconcave. Similarly, every convex function is quasiconvex.

Proposition

A concave function is quasiconcave. A convex function is quasiconvex.

Proof

Denote the function by f , and the (convex) set on which it is defined by S . Let a be a real number and let x and y be points in the upper level set P_a : $x \in P_a$ and $y \in P_a$. We need to show that P_a is convex. That is, we need to show that for every $\lambda \in [0,1]$ we have $(1 - \lambda)x + \lambda y \in P_a$.

First note that the set S on which f is defined is convex, so we have $(1 - \lambda)x + \lambda y \in S$, and thus f is defined at the point $(1 - \lambda)x + \lambda y$.

Now, the concavity of f implies that

$$f((1-\lambda)x + \lambda y) \geq (1-\lambda)f(x) + \lambda f(y).$$

Further, the fact that $x \in P_a$ means that $f(x) \geq a$, and the fact that $y \in P_a$ means that $f(y) \geq a$, so that

$$(1-\lambda)f(x) + \lambda f(y) \geq (1-\lambda)a + \lambda a = a.$$

Combining the last two inequalities, we have

$$f((1-\lambda)x + \lambda y) \geq a,$$

so that $(1-\lambda)x + \lambda y \in P_a$. Thus every upper level set is convex and hence f is quasiconcave.

The converse of this result is not true: a quasiconcave function may not be concave. Consider, for example, the function $f(x, y) = xy$ defined on the set of pairs of nonnegative real numbers. This function is quasiconcave (its upper level sets are the sets of points above rectangular hyperbolae), but is not concave (for example, $f(0, 0) = 0$, $f(1, 1) = 1$, and $f(2, 2) = 4$, so that $f((1/2)(0, 0) + (1/2)(2, 2)) = f(1, 1) = 1 < 2 = (1/2)f(0, 0) + (1/2)f(2, 2)$).

Some properties of quasiconcave functions are given in the following result. (You are asked to prove the first result in an exercise.)

Proposition

Let U be a function of many variables and let g be a function of a single variable.

- If U is quasiconcave and g is increasing then the function f defined by $f(x) = g(U(x))$ for all x is quasiconcave.
- If U is quasiconcave and g is decreasing then the function f defined by $f(x) = g(U(x))$ for all x is quasiconvex.

In an exercise you are asked to show that the sum of quasiconcave functions may not be quasiconcave.

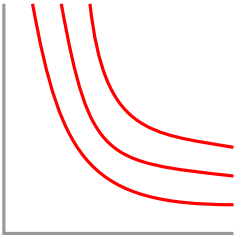
Why are economists interested in quasiconcavity?

The standard model of a decision-maker in economic theory consists of a set of alternatives and an ordering over these alternatives. The decision-maker is assumed to choose her favorite alternative—that is, an alternative with the property that no other alternative is higher in her ordering.

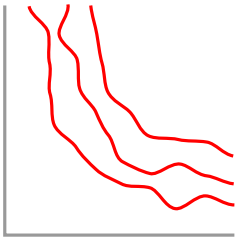
To facilitate the analysis of such a problem, we often work with a function that “represents” the ordering. Suppose, for example, that there are four alternatives, a , b , c , and d , and the decision-maker prefers a to b to c and regards c and d as equally desirable. This ordering is represented by the function U defined by $U(a) = 3$, $U(b) = 2$, and $U(c) = U(d) = 1$. It is represented also by many other functions—for example V defined by $V(a) = 100$, $V(b) = 0$, and $V(c) = V(d) = -1$. The numbers we assign to the alternatives are unimportant

except insofar as they are ordered in the same way that the decision-maker orders the alternatives. Thus any function W with $W(a) > W(b) > W(c) = W(d)$ represents the ordering.

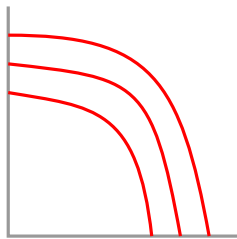
When the decision-maker is a consumer choosing between bundles of goods, we often assume that the level curves of the consumer's ordering—which we call “indifference curves”—look like this



and not like this



or like this

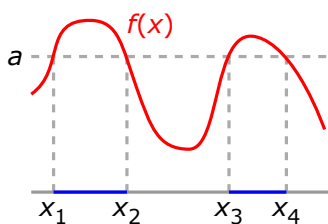


That is, we assume that every upper level set of the consumer's ordering is convex, which is equivalent to the condition that any function that represents the consumer's ordering is quasiconcave.

It makes no sense to impose a stronger condition, like concavity, on this function, because the only significant property of the function is the character of its level curves, not the specific numbers assigned to these curves.

Functions of a single variable

The definitions above apply to any function, including a function of a single variable. For a function of a single variable, an upper or lower level set is typically an interval of points, or a union of intervals. In the following figure, for example, the upper level set for the indicated value a —that is, the set of values of x for which $f(x) \geq a$ —is the union of the two blue intervals of values of x : the set of all values between x_1 and x_2 and between x_3 and x_4 .



By drawing some examples, you should be able to convince yourself that a function of a single variable is quasiconcave if and only if it is nondecreasing, or it is nonincreasing, or it is nondecreasing and then nonincreasing.

Proposition

A function f of a single variable defined on an interval I is quasiconcave if and only if one of the following conditions holds.

- f is nondecreasing
- f is nonincreasing
- there exists a number x^* such that f is nondecreasing on $\{x \in I: x \leq x^*\}$ and nonincreasing on $\{x \in I: x \geq x^*\}$.

Proof

Suppose that f satisfies one of the conditions and suppose that for some number a the points x_1 and x_2 in I are both members of the upper level set P_a . Then $f(x_1) \geq a$ and $f(x_2) \geq a$, so that $f(x) \geq a$ for every point x between x_1 and x_2 . Thus $x \in P_a$, so that P_a is convex and hence f is quasiconcave.

If f does not satisfy any of the conditions then we can find x_1, x_2 , and x_3 in I such that $x_1 < x_2 < x_3$ and $f(x_2) < \min\{f(x_1), f(x_3)\}$. Then the upper level set P_a for $a = \min\{f(x_1), f(x_3)\}$ includes x_1 and x_3 , but not x_2 , and hence is not convex, so that f is not quasiconcave.

Note that this result does **NOT** apply to functions of many variables!

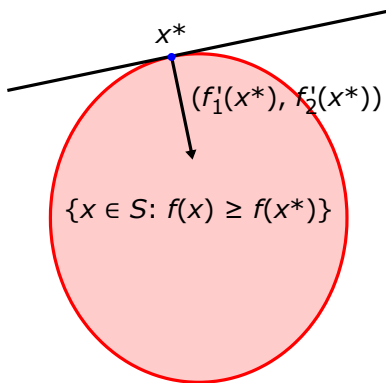
Differentiable functions

The fact that the graph of a differentiable concave function lies everywhere on or below all of its tangent planes leads to the result that the differentiable function f of many variables defined on the convex set S is concave if and only if

$$f(x) - f(x^*) \leq \sum_{i=1}^n f'_i(x^*) \cdot (x_i - x_i^*) \text{ for all } x \in S \text{ and } x^* \in S.$$

The fact that every upper level set of a differentiable quasiconcave function lies everywhere on one side of the tangent to the corresponding level curve, as illustrated in the following figure, implies a related characterization: a differentiable function f of many variables defined on the convex set S is quasiconcave if and only if

$$x \in S, x^* \in S, \text{ and } f(x) \geq f(x^*) \text{ imply } \sum_{i=1}^n f'_i(x^*) \cdot (x_i - x_i^*) \geq 0.$$



Like the characterization of differentiable concave functions, this characterization of differentiable quasiconcave functions is used in the theory of optimization to show that points that satisfy certain “first-order conditions” are global optimizers. The next result states the characterization, and the companion characterization for quasiconvex functions, precisely.

Proposition SOURCE

The differentiable function f of n variables defined on a convex set S is quasiconcave on S if and only if

$$x \in S, x^* \in S, \text{ and } f(x) \geq f(x^*) \text{ imply } \sum_{i=1}^n f'_i(x^*) \cdot (x_i - x_i^*) \geq 0$$

and is quasiconvex on S if and only if

$$x \in S, x^* \in S, \text{ and } f(x) \leq f(x^*) \text{ imply } \sum_{i=1}^n f'_i(x^*) \cdot (x_i - x_i^*) \leq 0.$$

Twice-differentiable functions

To determine whether a twice-differentiable function is quasiconcave or quasiconvex, we can examine the determinants of matrices known as "bordered Hessians".

Definition

Let f be a twice-differentiable function of n variables. For $r = 1, \dots, n$, the r th order **bordered Hessian** of f at the point x is the matrix

$$\begin{pmatrix} 0 & f'_1(x) & f'_2(x) & \dots & f'_r(x) \\ f'_1(x) & f''_{11}(x) & f''_{12}(x) & \dots & f''_{1r}(x) \\ f'_2(x) & f''_{21}(x) & f''_{22}(x) & \dots & f''_{2r}(x) \\ \dots & \dots & \dots & \dots & \dots \\ f'_r(x) & f''_{r1}(x) & f''_{r2}(x) & \dots & f''_{rr}(x) \end{pmatrix}.$$

Proposition SOURCE

Let f be a function of n variables with continuous partial derivatives and cross partial derivatives in an open convex set S and let D_r be the determinant of its r th order bordered Hessian.

- If f is quasiconcave then $D_1(x) \leq 0$, $D_2(x) \geq 0$, ..., $D_n(x) \leq 0$ if n is odd and $D_n(x) \geq 0$ if n is even, **for all** x in S . (Note that the first condition is automatically satisfied.)
- If f is quasiconvex then $D_k(x) \leq 0$ for $k = 1, \dots, n$ **for all** x in S . (Note that the first condition is automatically satisfied.)
- If $D_1(x) < 0$, $D_2(x) > 0$, ..., $D_n(x) < 0$ if n is odd and $D_n(x) > 0$ if n is even **for all** x in S then f is quasiconcave.
- If $D_k(x) < 0$ for $k = 1, \dots, n$ **for all** x in S then f is quasiconvex.

Another way to state this result is to say that

$$D_1(x) \leq 0, D_2(x) \geq 0, \dots, D_n(x) \leq 0 \text{ if } n \text{ is odd and } D_n(x) \geq 0 \text{ if } n \text{ is even, for all } x \text{ in } S$$

is a *necessary* condition for quasiconcavity, whereas

$$D_1(x) < 0, D_2(x) > 0, \dots, D_n(x) < 0 \text{ if } n \text{ is odd and } D_n(x) > 0 \text{ if } n \text{ is even for all } x \text{ in } S$$

is a *sufficient* condition, and similarly for quasiconvexity.

Note that the conditions don't cover all the possible cases, unlike the analogous result for concave functions. If, for example, $D_k(x) \leq 0$ for all k , for all x , but $D_r(x) = 0$ for some r and some x , then the result does not rule out the possibility that the function is quasiconvex, but it does not tell us that it is.

Example

Consider the function $f(x) = x^2$ for $x > 0$. We have $D_1(x) = -4x^2 < 0$ for all $x > 0$, so we deduce that this function is both quasiconcave and quasiconvex on the set $\{x: x > 0\}$.

Example

Consider the function $f(x) = x^2$ for $x \geq 0$. We have $D_1(0) = 0$, so this function does not satisfy the sufficient conditions for either quasiconcavity or quasiconvexity, although it is in fact both quasiconcave and quasiconvex (by a result for functions of a single variable, because it is nondecreasing).

Example

Consider the function $f(x_1, x_2) = x_1x_2$. For $x > 0$ the sufficient conditions for quasiconcavity are satisfied, while the necessary conditions for quasiconvexity are not. Thus the function is quasiconcave and not quasiconvex on the set $\{x: x > 0\}$. For $x \geq 0$ the sufficient conditions for quasiconcavity are not satisfied, but the necessary conditions are not violated. (The function is in fact quasiconcave on this domain.)

Another characterization of quasiconcavity

The following alternative characterization of a quasiconcave function (of any number of variables) is sometimes useful.

Proposition

The function f of many variables defined on a convex set S is quasiconcave if and only if for all $x \in S$ and $x' \in S$ such that $f(x) \geq f(x')$ we have

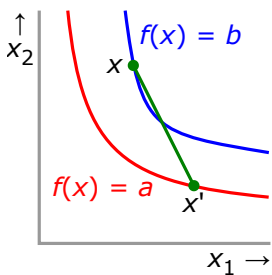
$$f((1-\lambda)x + \lambda x') \geq f(x') \text{ for all } \lambda \in [0, 1].$$

Proof

First suppose that f is quasiconcave. Let $x \in S$, $x' \in S$, and $f(x) \geq f(x')$. Then for $a = f(x')$ we have $x \in P_a$ and $x' \in P_a$. Because f is quasiconcave, P_a is convex, so $(1 - \lambda)x + \lambda x' \in P_a$ for all $\lambda \in [0, 1]$. Thus $f((1 - \lambda)x + \lambda x') \geq a = f(x')$ for all $\lambda \in [0, 1]$.

Now suppose that for all $x \in S$ and $x' \in S$ with $f(x) \geq f(x')$ we have $f((1-\lambda)x + \lambda x') \geq f(x')$ for all $\lambda \in [0, 1]$. For any number a , P_a is either empty, in which case it is convex, or consists of a single point, in which case it is convex, or contains two or more points. In the last case, choose $x \in P_a$ and $x' \in P_a$ with $f(x) \geq f(x')$. Then $f((1-\lambda)x + \lambda x') \geq f(x') \geq a$ (because $x' \in P_a$) for all $\lambda \in [0, 1]$. Thus $(1-\lambda)x + \lambda x' \in P_a$, so that P_a is convex and hence f is quasiconcave.

That is, a function is quasiconcave if and only if the line segment joining the points on two level curves lies nowhere below the level curve corresponding to the lower value of the function. This condition is illustrated in the following figure, in which $x = (x_1, x_2)$ and $b > a$: all points on the green line, joining x and x' , lie on or above the indifference curve corresponding to the smaller value of the function (a).



Strict quasiconcavity

This characterization of quasiconcavity motivates the following definition of a *strictly* quasiconcave function.

Definition

The function f of many variables defined on a convex set S is **strictly quasiconcave** if for all $x \in S$, all $x' \in S$ with $x' \neq x$, and all $\lambda \in (0,1)$ we have

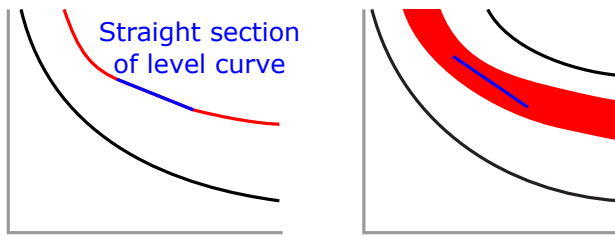
$$\text{if } f(x) \geq f(x') \text{ then } f((1-\lambda)x + \lambda x') > f(x').$$

That is, a function is strictly quasiconcave if every point, except the endpoints, on any line segment joining points on two level curves yields a higher value for the function than does any point on the level curve corresponding to the lower value of the function.

The definition says that a quasiconcave function of a single variable is strictly quasiconcave if its graph has no horizontal sections. For example, the function f defined by $f(x) = x^3$ is strictly quasiconcave.

For a function of two variables, it says that no level curve of a strictly quasiconcave function contains a line segment. (Take $x = x'$ in the definition.) Some level curves of two functions that are *not* strictly quasiconcave, although these level curves are consistent with the functions' being quasiconcave, are shown in the following

figures. In both cases, the red level curve contains a line segment, colored blue. (In the second figure, it does so because it is “thick”—see the [earlier example](#).)



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