Differentiation Under Integral Sign.

* Consider a continuous function $f(x,y)$ of two variables defined in a rectangle bounded by $x=a, x=b; y=c, y=d$. and the integral $\int_a^b f(x,y)\,dx$. Clearly the integral is a function of y. We write $\phi(y) = \int_a^b f(x,y)\,dx$.

It can be shown that $\phi(y)$ is a continuous function of y.

We further suppose that $f(x,y)$ possesses continuous first order partial derivative $f_y(x,y)$ w.r.t y. Under the assumption of the continuity of $f(x,y)$ and $f_y(x,y)$, it will be shown that

$$\phi'(y) = \frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial}{\partial y} f(x,y)\,dx,$$

the order of the two operations of differentiation and integration can be interchangeable.

Proof:- we have $\phi(y) = \int_a^b f(x,y)\,dx$

Let Δy denote a change in y. We have

$$\phi(y+\Delta y) = \int_a^b f(x, y+\Delta y)\,dx.$$

$$\therefore \phi(y+\Delta y) - \phi(y) = \int_a^b \left[f(x,y+\Delta y) - f(x,y)\right] dx$$

Using Lagrange's Mean Value theorem of Differential calculus, we obtain

$$\phi(y+\Delta y) - \phi(y) = \int_a^b \Delta y\, f_y(x, y+\theta\Delta y)\,dx, \quad 0<\theta<1$$

$$= \Delta y \int_a^b f_y(x, y+\theta\Delta y)\,dx$$

$$\therefore \frac{\phi(y+\Delta y)-\phi(y)}{\Delta y} = \int_a^b f_y(x,y+\theta\Delta y)\,dx$$

$$\therefore \frac{\phi(y+\Delta y)-\phi(y)}{\Delta y} = \int_a^b f_y(x,y)\,dx + \int_a^b \left[f_y(x,y+\theta\Delta y) - f_y(x,y)\right] dx. \quad \text{------ (1)}$$

Now since $f_y(x,y)$ is continuous. then for pre-assigned $\epsilon>0$, $\exists\ \delta>0$ such that $\left| f_y(x,y+\theta\Delta y) - f_y(x,y)\right| < \epsilon$, when $|\Delta y|<\delta$.

$$\therefore \left|\int_a^b \left\{ f_y(x,y+\theta\Delta y) - f_y(x,y)\right\} dx \right| < \epsilon\,(b-a), \text{ when } |\Delta y|<\delta.$$

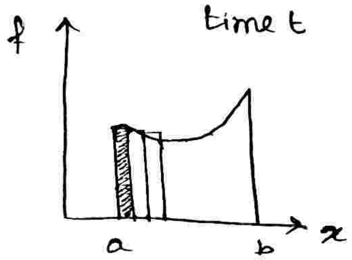
Then from (1) we have

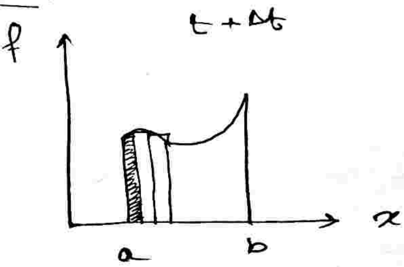
$$\left| \frac{\varphi(y+\Delta y)-\varphi(y)}{\Delta y} - \int_a^b f_y(x,y)\,dx \right| < \epsilon(b-a), \text{ when } |\Delta y| < \delta .$$

$$\therefore \lim_{\Delta y \to 0} \frac{\varphi(y+\Delta y)-\varphi(y)}{\Delta y} = \int_a^b f_y(x,y)\,dx$$

$$\therefore \quad \varphi'(y) = \int_a^b \frac{\partial}{\partial y} f(x,y)\,dx .$$

* Graphical Representation :-





$$I(t) = \int_a^b f(x,t)\,dx \qquad \frac{dI(t)}{dt} = \int_a^b \frac{\partial f(x,t)}{\partial t}.\,dx$$

* In the case of variable limits of integration :-

Now consider $\varphi(y) = \int_{a(y)}^{b(y)} f(x,y)\,dx$, where the limits $a(y), b(y)$ of integration are themselves functions of y. We assume that $a(y), b(y)$ possess continuous first order derivatives w.r.t y and $\varphi'(y) = \int_{a(y)}^{b(y)} f_y(x,y)\,dx + b'(y) f(b(y),y) - a'(y) f(a(y),y)$.

proof :- $\varphi(y) = \int_{a(y)}^{b(y)} f(x,y)\,dx$, $\varphi(y+\Delta y) = \int_{a(y+\Delta y)}^{b(y+\Delta y)} f(x, y+\Delta y)\,dx.$

$$\phi(y+\Delta y) = \int_{a(y+\Delta y)}^{b(y+\Delta y)} f(x, y+\Delta y)\, dx$$

$$= \int_{a(y+\Delta y)}^{a(y)} f(x, y+\Delta y)\, dx + \int_{a(y)}^{b(y)} f(x, y+\Delta y)\, dx + \int_{b(y)}^{b(y+\Delta y)} f(x, y+\Delta y)\, dx.$$

$$\therefore \phi(y+\Delta y) - \phi(y) = \int_{a(y)}^{b(y)} \left[f(x, y+\Delta y) - f(x,y)\right] dx + \int_{b(y)}^{b(y+\Delta y)} f(x, y+\Delta y)\, dx - \int_{a(y)}^{a(y+\Delta y)} f(x, y+\Delta y)\, dx$$

Now by Mean Value Theorem of Integral Calculus we have,

$$\int_{b(y)}^{b(y+\Delta y)} f(x, y+\Delta y)\, dx = \left[b(y+\Delta y) - b(y)\right] f(\xi, y+\Delta y).$$

and $$\int_{a(y)}^{a(y+\Delta y)} f(x, y+\Delta y)\, dx = \left[a(y+\Delta y) - a(y)\right] f(\eta, y+\Delta y),$$

wher ξ lies between $b(y)$, $b(y+\Delta y)$, η lies between $a(y)$, $a(y+\Delta y)$.

MVT for integral calculus
If $f(x)$ is continuous on $[a,b]$, then $\exists$ a number ξ, between a, b s.t. $\int_a^b f(x)\,dx = (b-a) f(\xi)$.

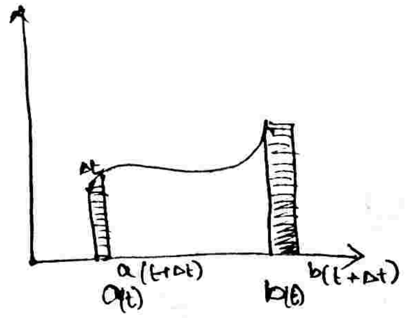
$$\therefore \phi(y+\Delta y) - \phi(y) = \int_{a(y)}^{b(y)} \Delta y\, f_y(x, y+\theta\Delta y)\, dx + \left[b(y+\Delta y) - b(y)\right] f(\xi, y+\Delta y)$$
$$- \left[a(y+\Delta y) - a(y)\right] f(\eta, y+\Delta y)$$

$$\therefore \frac{\phi(y+\Delta y) - \phi(y)}{\Delta y} = \int_{a(y)}^{b(y)} f_y(x, y+\theta\Delta y)\, dx + \frac{b(y+\Delta y) - b(y)}{\Delta y} f(\xi, y+\Delta y)$$
$$- \frac{a(y+\Delta y) - a(y)}{\Delta y} f(\eta, y+\Delta y) \quad \text{--- (2)}$$

Taking $\Delta y \to 0$ in (2)

$$\phi'(y) = \int_{a(y)}^{b(y)} f_y(x,y)\, dx + b'(y)\left[f(b(y), y)\right] - a'(y)\left[f(a(y), y)\right]$$

Graphical representation :-



$$\frac{d}{dt}\left(\int_{a(t)}^{b(t)} f(x,t)\, dx \right) = f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt} + \int_a^b \frac{\partial f(x,t)}{\partial t}\, dx.$$

* Evaluate $\int_0^a \frac{\log(1+ax)}{1+x^2}\, dx$ and show that

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = \frac{\pi}{8}\log 2.$$

Soln:- we write $\phi(a) = \int_0^a \frac{\log(1+ax)}{1+x^2}\, dx$ ——— (1)

$$\therefore \phi'(a) = \int_0^a \frac{\partial}{\partial a}\left(\frac{\log(1+ax)}{1+x^2}\right) dx + 1 \cdot \frac{\log(1+a^2)}{1+a^2}$$

$$= \int_0^a \frac{x}{(1+ax)(1+x^2)}\, dx + \frac{\log(1+a^2)}{1+a^2}$$

Now $\frac{x}{(1+ax)(1+x^2)} = -\frac{a}{(1+a^2)(1+ax)} + \frac{x+a}{(1+a^2)(1+x^2)}$

$$\therefore \int_0^a \frac{x}{(1+ax)(1+x^2)}\, dx = -\frac{1}{1+a^2}\left|\log(1+ax)\right|_0^a + \frac{1}{2(1+a^2)}\log(1+x^2)\Big|_0^a + \frac{a\tan^{-1}x}{1+a^2}\Big|_0^a$$

$$= -\frac{1}{2(1+a^2)}\log(1+a^2) + \frac{a}{1+a^2}\tan^{-1}a$$

$$\therefore \phi'(a) = \frac{1}{2(1+a^2)}\log(1+a^2) + \frac{a}{1+a^2}\tan^{-1}a.$$

Integrating, we get

$$\varphi(a) = \frac{1}{2}\left[\int \frac{1}{1+a^2}\log(1+a^2)\,da + \int \frac{a}{1+a^2}\tan^{-1}a\,da\right]$$

$$= \frac{1}{2}\left[\tan^{-1}a\log(1+a^2) - \int \frac{2a}{1+a^2}\tan^{-1}a\,da\right]$$

$$+ \int \frac{a}{1+a^2}\tan^{-1}a\,da + c$$

$$= \frac{1}{2}\tan^{-1}a\log(1+a^2) + C \quad \text{------} \quad (2)$$

From (1), we see that $\varphi(0) = 0$. Putting $a=0$ in (2), we get $C=0$

$\therefore \varphi(a) = \frac{1}{2}\tan^{-1}a\log(1+a^2)$.

From this, taking $a=1$, we get

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \varphi(1) = \frac{1}{2}\tan^{-1}(1)\log 2 = \frac{\pi}{8}\log 2.$$

* Show that $\int_0^\infty e^{-x^2}\cos\alpha x\,dx = \frac{1}{2}\sqrt{\pi}\,e^{-\frac{1}{4}\alpha^2}$.

$$\varphi(\alpha) = \int_0^\infty e^{-x^2}\cos\alpha x\,dx \quad \text{-------} \quad (1)$$

$$\therefore \varphi'(\alpha) = -\int_0^\infty x e^{-x^2}\sin\alpha x\,dx$$

Integrating by parts, we have

$$\varphi'(\alpha) = \frac{1}{2}e^{-x^2}\sin\alpha x\Big|_0^\infty - \frac{\alpha}{2}\int_0^\infty e^{-x^2}\cos\alpha x\,dx$$

$$= 0 - \frac{\alpha}{2}\varphi(\alpha) = -\frac{\alpha}{2}\varphi(\alpha)$$

$$\therefore \frac{\varphi'(\alpha)}{\varphi(\alpha)} = -\frac{\alpha}{2}.$$

Integrating, we get

$$\log\varphi(\alpha) = -\frac{1}{4}\alpha^2 + c_1 \Rightarrow \varphi(\alpha) = ce^{-\frac{1}{4}\alpha^2} \quad \text{----} \quad (2)$$

Putting $\alpha=0$ in (1), we get

$$\varphi(0) = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$x^2 = u$

$2x\,dx = du$

$\frac{1}{2}\int_0^\infty e^{-u}\cdot u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}\ [\because \Gamma(\frac{1}{2}) = \sqrt{\pi}]$

Putting $\alpha=0$ in (2), we get

$\varphi(0) = C$.

$$\therefore \varphi(\alpha) = \frac{1}{2}\sqrt{\pi}\,e^{-\frac{1}{4}\alpha^2}.$$