Position measurement – gamma ray microscope thought experiment

If one wants to be clear about what is meant by "position of an object," for example of an electron..., then one has to specify definite experiments by which the "position of an electron" can be measured; otherwise this term has no meaning at all.

--Heisenberg, 1927

Heisenberg pictured a microscope that obtains very high resolution by using high-energy gamma rays for illumination. No such microscope exists at present, but it could be constructed in principle. Using this microscope, Heisenberg imagined to see an electron and also to measure its position. He found that the electron's position and momentum did indeed obey the uncertainty relation he had derived mathematically.

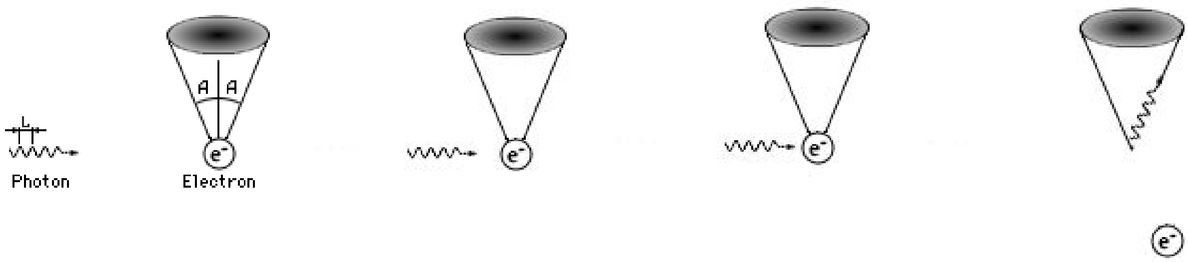

Fig. 1(a), 1(b), 1(c), 1(d).

A free electron sits directly beneath the center of the microscope's lens (see the picture above). The circular lens forms a cone of angle 2A from the electron. The electron is then illuminated from the left by gamma rays--high energy light which has the shortest wavelength. This yields the highest resolution. According to a principle of wave optics, the microscope can resolve objects to a size of Δx, which is related to the wavelength L of the gamma ray, by the expression:

$$\Delta x = \frac{L}{\sin A}$$

It appears that by making L small, that is why we choose gamma-ray, and by making $\sin A$ large, Δx can be made as small as desired. But, according to uncertainty principle, we can do so only at the expense of our knowledge of x-component of electron momentum.

However, in quantum mechanics, where a light wave can act like a particle (photon), a gamma ray striking an electron gives it a kick (Compton Effect). At the moment the light is diffracted by the electron into the microscope lens, the electron is thrust to the right (Fig.1(d)). To be observed by the microscope, the gamma ray must be scattered into any angle within the cone of angle 2A. In quantum mechanics, the gamma ray carries momentum, as if it were a particle. The total momentum p is related to the wavelength by the formula

$$p = \frac{h}{L} \ ,$$

where h is the Planck's constant.

In the extreme case of diffraction of the gamma ray to the right edge of the lens, the total momentum in the x-direction would be the sum of the electron's momentum p'_x in the x-direction and the gamma ray's momentum in the x-direction:

$$p'_x + \frac{(h\sin A)}{L'} \ ,$$

where L' is the wavelength of the deflected gamma ray.

In the other extreme, the observed gamma ray recoils backward, just hitting the left edge of the lens. In this case, the total momentum in the x direction is:

$$p''_x - \frac{(h\sin A)}{L''} \ ,$$

where L'' is the wavelength of the deflected gamma ray.

The final x momentum in each case must equal the initial x momentum, since momentum is never lost (it is *conserved*). Therefore, the final x momenta are equal to each other:

$$p'_x + \frac{(h\sin A)}{L'} = p''_x - \frac{(h\sin A)}{L''}$$

If A is small, then the wavelengths are approximately the same, L' ~ L'' ~ L. So we have

$$p''_x - p'_x = \Delta p_x \approx \frac{(2h\sin A)}{L}$$

Since $\sin A = \frac{L}{\Delta x}$, we obtain a reciprocal relationship between the minimum uncertainty in the measured position, Δx, of the electron along the x-axis and the uncertainty in its momentum, Δp_x, in the x-direction:

$$\Delta p_x \sim \frac{2h}{\Delta x} \quad \text{or} \quad \Delta x \Delta p_x \sim 4\pi\hbar \text{ (Heisenberg's uncertainty relation)}$$

References:

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3. Bransden, B.H., & Joachain, C.J. (2004).Quantum Mechanics, 2nd ed., Pearson Education: India. Pp 70-72.