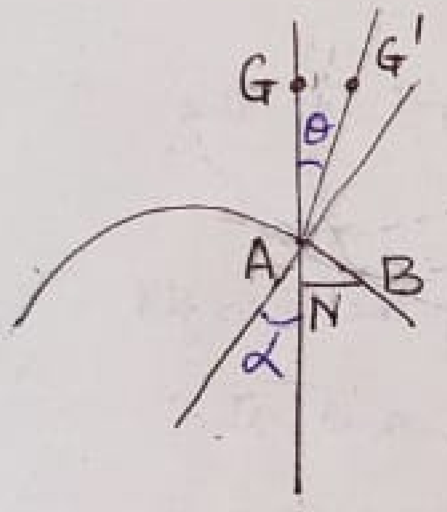
ROCKING STONES

Previously we have discussed the stability of equilibrium of one body resting on another. To do so we made a number of assumptions.

These were that the portions of the two bodies in contact are spherical and that their point of contact is the highest point of the lower body. Now we remove both these restriction to discuss the stability.

We state the problem as the following.



A perfectly rough heavy body rests in equilibrium on a fixed body at the point A. The angle made with the vertical by the common normal at the point of contact A is given by α. For equilibrium the centre of gravity G of the upper body must lie on the vertical at A. Also G is at the height h above A. ie $AG = h$. The equilibrium is stable or unstable according as $\frac{\cos\alpha}{h} \gtrless \frac{1}{\rho_1} + \frac{1}{\rho_2}$, where ρ_1 and ρ_2 are the radii of curvature at A of the curves of intersection in which the two bodies are cut by the vertical plane of symmetry, in which the displacement occurs.

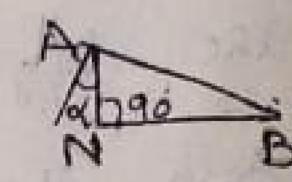
Proof Let the upper body rolls on the lower body through a small angle θ is such that, B is the new ~~position poi~~ point of contact.

Let arc $AB = s$, then we have

$$\theta = \frac{s}{\rho_1} + \frac{s}{\rho_2} \quad \text{ie} \quad \theta = s\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \quad --(1)$$

where small quantities of second order has been neglected and s is a small quantity of first order. Relation (i) can easily be obtained by using geometrical facts.

Since A is the instantaneous centre of rotation of the upper body when it rolls, G moves horizontally to G' and its horizontal displacement GG' is equal to $h\theta$.



Next we draw BN perpendicular to the vertical AG. Then the equilibrium is stable or unstable according as $GG' \lesseqgtr BN$.

Now, the line of action of the weight of the upper body is the vertical through G'.

When $GG' < BN$, the line of action of the weight of the upper body passes to the left of B and the moment of this weight about B will tend to move the upper body back to its original position of equilibrium.

The reverse will happen when $GG' > BN$.

When B is very close to A, AB becomes the tangent at A and $\angle BAN = (90° - \alpha)$.

In $\triangle BAN$, $\sin \angle BAN = \dfrac{BN}{AB}$.

or, $BN = AB \sin \angle BAN = s \sin(90° - \alpha) = s\cos\alpha$

Thus $GG' \lesseqgtr BN$ implies $h\theta \lesseqgtr s\cos\alpha$.

$\therefore$ Using (i) we get

$$h s \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \lesseqgtr s\cos\alpha \quad \text{------- (ii)}$$

or, $$\frac{\cos\alpha}{h} \gtreqless \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Thus the equilibrium is stable or unstable according as

$$\frac{\cos\alpha}{h} \gtreqless \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Specifically, if the upper body rests at the highest point of the lower body, we have $\alpha = 0$.
Hence the condition of stability or unstability becomes

$$\frac{1}{h} \gtrless \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

But if $\dfrac{\cos\alpha}{h} = \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$,

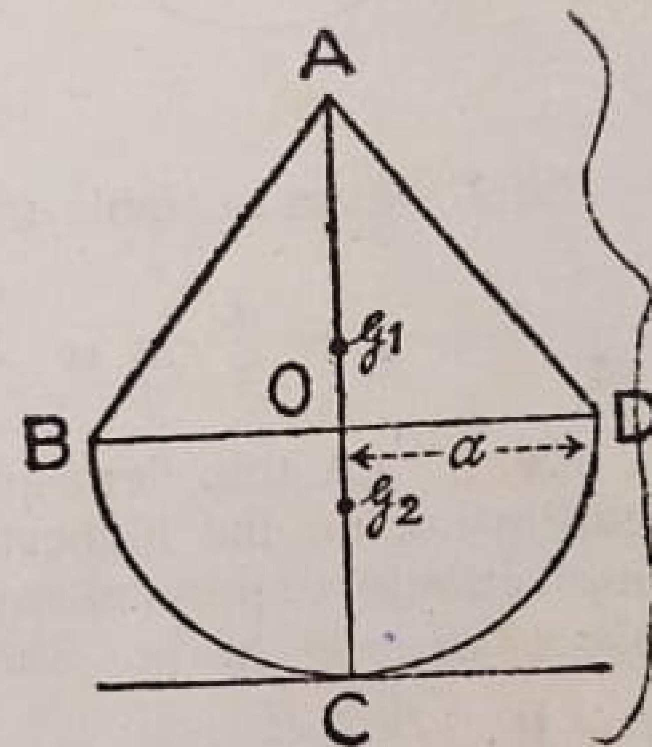
the equilibrium is said to be neutral to a first approximation, for to this order, the new position of the centre of gravity of the upper body will lie in the vertical through the new point of contact.

4.9. Problem.

In most of the problems, gravity is th only external force and the system has one degree of freedom. Method of procedure to be adopted for solving such problems has been indicated in Art. 4.5. In some problems results proved in Art. 4.6 and the more general result proved in Art. 4.8 are directly applicable. Some problems of different types will now be solved for purpose of illustration.

Worked Examples

Ex. 1. A body, consisting of a cone and a hemisphere on the same hase, rests on a rough horizontal table, the hemisphere being in contact with the table ; show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.



Let ABD be the cone and BCD the hemisphere. Let AOC be the common axis on which lie the c. g.'s G_1, G_2 of the cone and the hemisphere. $\therefore$ G, the c. g. of the combined body is on AOC. If G does not coincide with O, the centre of the sphere, then in position of equilibrium AOC is vertical and C is the point of contact with the horizontal table. Let a = radius of hemisphere and h = height of cone.

$$CG_2 = \frac{5}{8}a, \text{ and } CG_1 = a + \frac{h}{4}.$$

$$\text{Volume of hemisphere} = \frac{2}{3}\pi a^3, \quad \text{volume of cone} = \frac{1}{3}\pi a^2 h.$$

$$\therefore \; CG = \frac{\frac{2}{3}\pi a^3 \times \frac{5}{8}a + \frac{1}{3}\pi a^2 h\left(a + \frac{h}{4}\right)}{\frac{2}{3}\pi a^3 + \frac{1}{3}\pi a^2 h} = \frac{5a^2 + 4ah + h^2}{4(2a + h)}$$

$$\text{Equilibrium is stable if } \frac{1}{CG} > \frac{1}{a} \quad \text{or,} \quad \frac{4(2a+h)}{5a^2 + 4ah + h^2} > \frac{1}{a}$$

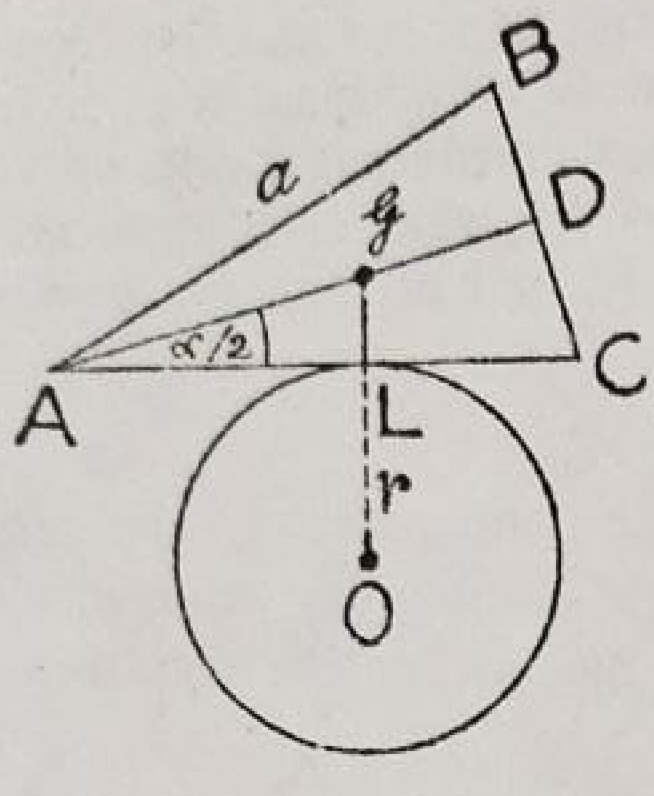
$$\text{or,} \quad 3a^2 > h^2 \quad \text{or} \quad h < a\sqrt{3}.$$

$\therefore$ greatest value of h for stable equilibrium $= a\sqrt{3}$.

If G coincides with O, which happens $h = a\sqrt{3}$, the system will be in equilibrium with any point of BCD in contact with the table.

Ex. 2. A limina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere ; show that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha < \frac{3r}{a}$, where a is one of the equal sides of the triangle.

Let ABC be the triangle where $AC = AB = a$ and $\angle BAC = \alpha$. The triangle rests at the highest point L of the sphere with side AC in contact.



Let $BD = DC$, then the c. g. G of the triangle is on AD, and also it must be on the vertical OL at L. $\therefore$ G is the point of intersection of OL and AD and

$$AG = \frac{2}{3} AD = \frac{2}{3} . AC \cos \frac{\alpha}{2}$$

$$= \frac{2}{3} a \cos \frac{\alpha}{2} \quad \text{and} \quad h = LG = AG \sin \frac{\alpha}{2}$$

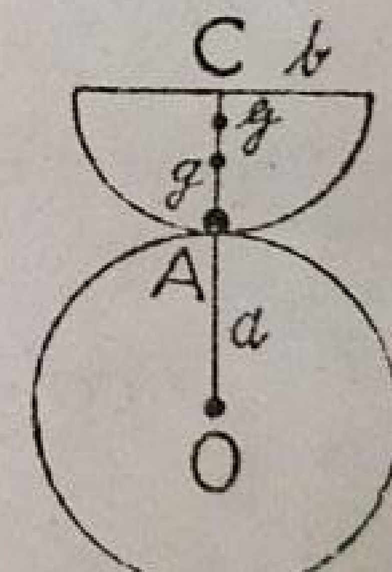
$$= \frac{2}{3} a \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = \frac{a}{3} \sin \alpha.$$

Equilibrium is stable if $\frac{1}{h} > \frac{1}{r}$

$$or, \quad r > \frac{a}{3} \sin \alpha \quad or, \sin \alpha > \frac{3r}{a}.$$

Ex. 3. A thin hemispherical bowl, of radius b and weight W, rests in equilibrium on the highest point of a fixed sphere, of radius a, which is rough enough to prevent any sliding. Inside the bowl is placed a small smooth sphere of weight ω. Show that the equilibrium is not stable unless $\omega < W . \frac{a - b}{2b}$.

Let A be the highest point of the fixed sphere of centre O and let C be the centre of the plane face of the bowl. Then OAC is the common normal at A and it is vertical. In the position of equilibrium, g, the c. g. of the bowl, must lie on the vertical at A. $\therefore$ $OAgC$ is one straight line and A is the lowest point of the bowl ; we have, $Cg = \frac{b}{2}$. The weight ω always occupies the lowest position in the bowl and its line of action always passes throug C. $\therefore$, its effect will remain unalter if it be fixed to the bowl at C. If G be the c. g. of the combined body, the bowl and the weight ω fixed at C, then



$$h = AG = \frac{W . \frac{b}{2} + \omega . b}{W + \omega} . \quad \text{For stable equilibrium} \quad \frac{1}{h} > \frac{1}{a} + \frac{1}{b}$$

$$or, \quad \frac{2(W + \omega)}{b(W + 2\omega)} > \frac{a + b}{ab}$$

$$or, \quad 2a(W + \omega) > (a + b)(W + 2\omega)$$

$$or, \quad \omega < \frac{W(a - b)}{2b}.$$