FIRST ORDER LINEAR DIFFERENTIAL EQUATION

Economic variables are often sensitive to time changes. The rate of change of economic variables in response to time variations can be known by the method of simple differentiation. For example, let $y = f(t)$, where t is time. Then the rate of change of y over time is given by $\frac{dy}{dt}$. Thus once any functional relation is given, the rate of change of the dependent variable with respect to any change in the independent variable can be known by the derivatives. But what will happen if we like to go other way round? If we are given the rate of change of a variable with respect to any change in other variable/variables, would it be possible for us to get back the original functional form from that? The answer is: yes. We can obtain it by solving differential equations. In the analysis of differential equations the time variable is assumed to be continuous. This means that the time bound variable changes at each point of time.

// Order refers to the highest order of the derivatives or differentials appearing in the differential equation; thus a first-order differential equation can contain only the first derivatives say, dy/dt.

// The first derivative dy/dt is the only one that can appear in a first order differential equation, but it may enter in various powers: dy/dt, $(dy/dt)^2$, or $(dy/dt)^3$. The highest power attained by the derivative in the equation is referred to as the degree of the differential equation. In case the derivative dy/dt appears only in the first degree, and so does the dependent variable y, and furthermore, no product of the form $y\left(\frac{dy}{dt}\right)$ occurs, then the equation is said to be linear.

Thus a first-order linear differential equation will generally take the form

$$\frac{dy}{dt} + u(t)\,y = w(t) \quad \cdots\cdots \quad (1)$$

where u and w are two functions of t, as is y.

When the function u (the coefficient of the dependent variable y) is a constant, and when the function w is a constant additive term, (1) reduces to the special case of a first-order linear differential equation with constant coefficient and constant term.

CONSTANT COEFFICIENT AND CONSTANT TERM

* The Homogenous Case

If u and w are constant functions, and if w happens to be identically zero, (1) becomes

$$\frac{dy}{dt} + ay = 0 \qquad (2)$$

where 'a' is some constant. This differential equation is said to be homogenous on account of the zero constant term.

General Solution $y(t) = Ae^{-at}$

$\therefore \frac{dy}{dt} = -a\,Ae^{-at}$

Definite Solution $y(t) = y(0)\,e^{-at}$

* The Non-homogeneous Case

$$\frac{dy}{dt} + ay = b$$

The solution of this nonhomogeneous equation will consist of the sum of two terms, one of which is called the complementary function (y_c) and the other known as the particular integral (y_p).

Complementary function

The reduced equation

$$\frac{dy}{dt} + ay = 0$$

$$\therefore y_c = Ae^{-at}$$

Particular Integral

$$y = k \text{ (constant)} \qquad \frac{dy}{dt} = 0$$

$$\therefore 0 + ka = b \Rightarrow k = b/a$$

$$\therefore y(t) = y_c + y_p$$

$$= Ae^{-at} + b/a \quad [\text{when } a \neq 0]$$

Now, $t = 0$

$$y(0) = A + b/a \Rightarrow A = [y(0) - b/a]$$

$$\therefore \boxed{y_t = \left[y(0) - \frac{b}{a}\right]e^{-at} + b/a} \quad [\text{when } a \neq 0]$$

When $a = 0$,

we consider $y = kt \Rightarrow \frac{dy}{dt} = k$

$\therefore k + a.kt = b$

$\therefore k = b$ [as $a = 0$]

$\therefore y_p = bt$

$\therefore y(t) = y_c + y_p$

$= Ae^{-at} + bt$

When $t = 0$

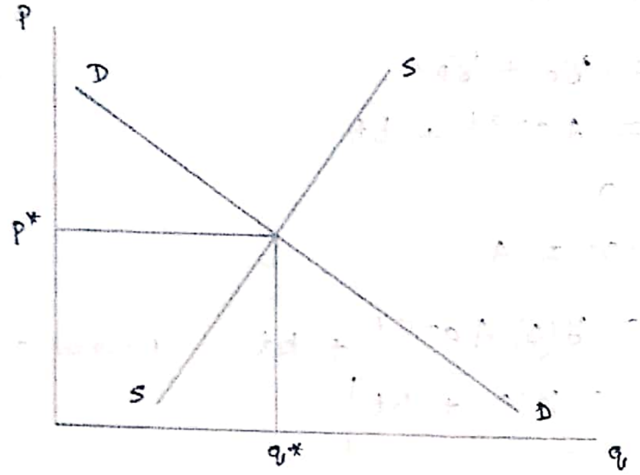
$\therefore y(0) = A$

$\therefore y(t) = y(0) Ae^{-at} + bt$ when $a = 0$

$\therefore \boxed{y(t) = y(0) + bt}$

DYNAMICS OF MARKET PRICE (A Walrasian Price Adjustment Model):-

In a Competitive market, price is determined by supply and demand. Following figure shows a typical supply and demand diagram with equilibrium price at p^* and the equilibrium quantity at q^*.



Is the point (p^*, q^*) a stable equilibrium? If it is, then even if the market is temporarily out of equilibrium for some reason, price and quantity will return to their equilibrium values. If it is an unstable equilibrium, then price and quantity will not return to the equilibrium if the market is ever put into disequilibrium.

How can we determine in general whether an equilibrium is stable or not? An obvious way is to analyse the dynamics of the model of supply and demand.

To answer this question, we must first find the time path $P(t)$. First order linear differential equation can be use to trace the time path of the price movement to the equilibrium and from the equilibrium price. i.e. Convergence or divergence in the demand supply model with respect to time. To derive $P(t)$, it requires a specific pattern of price change to be prescribed first. In general, price changes are governed by the relative strength of the demand and supply forces in the market. Let us assume, for the sake of simplicity, that the rate of price change (with respect to time) at any moment is always directly proportional to the excess demand $(Q_d - Q_s)$, prevailing at that moment. Such a pattern of change can be expressed symbolically as

$$\frac{dP}{dt} = j(Q_d - Q_s) \quad (j>0) \quad \text{.....(1)}$$

[Q_d = Quantity demanded; Q_s = Quantity supplied]

where j represents a (Constant) adjustment Coefficient

Equation (1) states that if $Q_d = Q_s$

then $\frac{dP}{dt} = 0$

Let the demand function for a particular Commodity

$$Q_d = \alpha - \beta P \quad (\alpha, \beta > 0) \quad \ldots\ldots\ldots (2)$$

and supply function

$$Q_s = -\gamma + \delta P \; (\gamma, \delta > 0) \quad \ldots\ldots\ldots (3)$$

Setting $Q_d = Q_s$ and solving for the equilibrium price P^*

$$Q_d = Q_s$$

or, $\alpha - \beta P = -\gamma + \delta P$

or, $P(\beta + \delta) = \alpha + \gamma$

$$\therefore P^* = \frac{\alpha + \gamma}{\beta + \delta}$$

If initial price $P(0) = P^* \Rightarrow$ the market will clearly be in equilibrium already, and no dynamic analysis will be needed. But if $P(0) \neq P^*$, then P^* is attainable only after a due process of adjustment, during which not only will price change over time but Q_d and Q_s, being functions of P, must change over time as well. In this light, then, the price and quantity variables can all be taken to be function of time. The equilibrium condition $P(0) = P^*$ is stable, in the static sense, if any deviation from this equilibrium sets market forces so as to restore the equilibrium position. But in the dynamic sense, this equilibrium is stable if the time path of price converges to this equilibrium, i.e $P(t) \to P^*$, as $t \to \infty$.

Now Putting the values of Q_d and Q_s in equation (1), we get

$$\frac{dP}{dt} = j(\alpha - \beta P + \gamma - \delta P)$$

or, $\frac{dP}{dt} = j(\alpha + \gamma) - j(\beta + \delta)P$

$$\therefore \frac{dP}{dt} + j(\beta + \delta)P = j(\alpha + \gamma) \quad \ldots\ldots\ldots (4)$$

This is a first order non-homogeneous differential equation.

Particular integral:-

Let $P = P^*$ be the intertemporal equilibrium price.

$$\therefore \frac{dP}{dt} = 0$$

$$\therefore \quad j(\beta+\delta)P = j(\alpha+\gamma)$$

$$\therefore \; P^* = \frac{\alpha+\gamma}{\beta+\delta}$$

Complementary Function

Now equation (4) is reduced to the homogeneous form

$$\frac{dP}{dt} + j(\beta+\delta)P = 0 \quad \cdots\cdots (5)$$

Let $P(t) = Ae^{-at}$ be a trial solution of equation (5)

$$\therefore \frac{dP}{dt} = -a.Ae^{-at}$$

From equation (5) we get

$$-a.Ae^{-at} + j(\beta+\delta).Ae^{-at} = 0$$

$$\therefore \; Ae^{-at}\left[-a + j(\beta+\delta)\right] = 0$$

$$\because \; Ae^{-at} \neq 0$$

$$\therefore \; -a + j(\beta+\delta) = 0$$

$$\therefore \; a = j(\beta+\delta)$$

$$\therefore \; P_c = Ae^{-j(\beta+\delta)t}$$

The sum of the complementary function and the particular integral then constitute the general solution of the equation (4).

$$P(t) = Ae^{-j(\beta+\delta)t} + \frac{\alpha+\gamma}{\beta+\delta} \quad \cdots\cdots (6)$$

We assume that P takes the P(0) when $t=0$. Then, by setting $t=0$ in equation (6), we find that

$$P(0) = A + \frac{\alpha+\gamma}{\beta+\delta}$$

$$\therefore \; A = P(0) - \frac{\alpha+\gamma}{\beta+\delta}$$

$\therefore$ The time path of price is

$$P(t) = \left[P(0) - \frac{\alpha+\gamma}{\beta+\delta}\right]e^{-j(\beta+\delta)t} + \frac{\alpha+\gamma}{\beta+\delta}$$

$$= \left[P(0) - \frac{\alpha+\gamma}{\beta+\delta}\right]e^{-Kt} + \frac{\alpha+\gamma}{\beta+\delta} \quad \cdots\cdots (7)$$

$$= \left[P(0) - P^*\right]e^{-Kt} + P^* \quad \left[\text{where } j(\beta+\delta) = K\right]$$

The Dynamic stability of Equilibrium

Let us now see whether this equilibrium is dynamically stable or not, i.e whether $P(t) \to P^*$ as $t \to \infty$

But it depends on the first term on the right of equation (7),

i.e. $[P(0) - P^*] e^{-kt} \to 0$ as $t \to \infty$

Since $P(0)$ and P^* are both constant, then the key factor will be the exponential expression e^{-kt}. In view of the fact that $k > 0$, that expression does tend to zero. as $t \to \infty$. Now if time path of price, $P(t)$ converges to the level P^*, then the equilibrium is said to be dynamically stable.

Here the dynamic stability of $P(t)$ depends on the relative magnitudes of $P(0)$ and P^*, and the equation (7) encompasses three possible cases.

Case - I $P(0) = P^* \Rightarrow P(t) = P^*$

$\Rightarrow$ the time path of price will be the horizontal straight line in the following figure.

Case - II $P(0) > P^*$

$\therefore [P(0) - P^*] > 0$ but e^{-kt} decreases as t increases

Thus $[P(0) - P^*] e^{-kt}$ decreases as t increases,

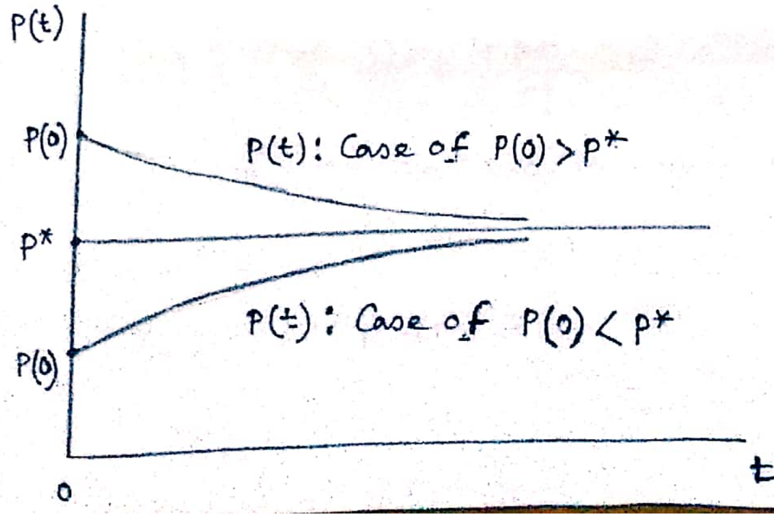
$\Rightarrow$ the time path will approach the equilibrium level P^* from above, as illustrated by the top curve in the following figure.

Case - III $P(0) < P^*$

$\therefore [P(0) - P^*] < 0$ but e^{-kt} decreases as t increases

Thus $[P(0) - P^*] e^{-kt}$ increases as t increases.

$\Rightarrow$ the time path will approach the equilibrium level P^* from below, as illustrated by the bottom curve of the same figure.



Since we have $K > 0$

then $P(t)$ Converges P^* as $t \to \infty$

and the equilibrium is dynamically stable. Thus dynamic stability requires the asymptotic vanishing of the Complementary function as t becomes infinity.

Q:- Let the demand and supply be

$$Q_d = \alpha - \beta P + \sigma \frac{dP}{dt}$$

$$Q_s = -\gamma + \delta P \qquad [\alpha, \beta, \gamma, \delta > 0]$$

(a) Assume that the rate of change of price over time is directly proportional to the excess demand, find the time path $P(t)$. [general solution]

(b) What is the intertemporal equilibrium price?

(c) What is the restriction on the parameter σ would ensure dynamic stability?