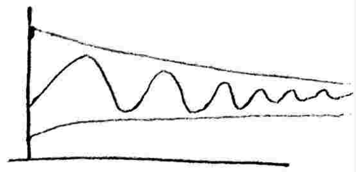
* Cauchy's criterion :-

Cauchy's general principle of convergence :-

A necessary and sufficient condition for the convergence of a seqn $\{u_n\}$ is that for a pre-assigned $\epsilon > 0$, $\exists$ a $m \in \mathbb{N}$ s.t

$$|u_{n+p} - u_n| < \epsilon \quad \forall \; n \geq m \text{ and for } p = 1, 2, 3, \ldots$$

1. Use Cauchy's principle of convergence to prove that the sequence $\left\{\frac{n}{n+1}\right\}$ is convergent.



Let $u_n = \frac{n}{n+1}$. Let p be a natural number.

Then $u_{n+p} = \frac{n+p}{n+p+1}$

$$|u_{n+p} - u_n| = \left| \frac{n+p}{n+p+1} - \frac{n}{n+1} \right| = \frac{p}{(n+p+1)(n+1)} < \frac{1}{n+1} < \frac{1}{n} \; \forall \; p$$

Since $\frac{p}{n+p+1} < 1 \; \forall \; p$.

Let $\epsilon > 0$. Then $\frac{1}{n} < \epsilon \; \forall \; n > \frac{1}{\epsilon}$.

Let $m = \left[\frac{1}{\epsilon}\right] + 1$. Then m is a natural number and

$|u_{n+p} - u_n| < \epsilon \; \forall \; n \geq m$ and $p = 1, 2, 3, \ldots$

$\therefore \{u_n\}$ is convergent.

2. Prove that $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not convergent.

Let p be a natural number.

$$|u_{n+p} - u_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}$$

Let us choose $n = m$ and $p = m$.

Then
$$|u_{2m} - u_m| = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m}$$
$$> \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m}$$
$$= \frac{1}{2}.$$

If we choose $\epsilon = \frac{1}{2}$ then $\nexists \; K \in \mathbb{N}$ s.t

$|u_{n+p} - u_n| < \epsilon \; \forall \; n \geq K$ for every natural number p.

$\Rightarrow$ Cauchy's condition is not satisfied by the seqn

Therefore seqn $\{u_n\}$ is not convergent.

* **Cauchy Sequence:-** A seqn $\{u_n\}$ is said to be a Cauchy seqn if for a pre-assigned positive ϵ, $\exists$ a natural number K such that

$|u_m - u_n| < \epsilon \quad \forall\ m, n \geq K.$

NOTE:- Replacing m by $n+p$, where $p = 1, 2, 3, \ldots$ the above condition can be equivalently stated as

$|u_{n+p} - u_n| < \epsilon \quad \forall\ n \geq K$ and $p = 1, 2, 3, \ldots$

Thm:- A cauchy seqn of real numbers is convergent.

Let $\{u_n\}$ be a Cauchy seqn. First we prove that the seqn $\{u_n\}$ is bounded.

Let $\epsilon = 1$. Then there exists a natural number K such that

$|u_m - u_n| < 1 \quad \forall\ m, n \geq K.$

$\Rightarrow \quad |u_K - u_n| < 1 \quad \forall\ n \geq K.$

$\Rightarrow \quad u_K - 1 < u_n < u_K + 1 \quad \forall\ n \geq K.$

Let $B = \max\{u_1, u_2, \ldots, u_{K-1}, u_K + 1\}$

$b = \min\{u_1, u_2, \ldots, u_{K-1}, u_K - 1\}$.

Then $b \leq u_n \leq B \ \forall\ n \in \mathbb{N}$ and this proves that the seqn $\{u_n\}$ is bounded.

By Bolzano-Weierstrass theorem $\{u_n\}$ has a convergent subseqn. Then let l be the limit of that convergent subsequence.

l is a subsequential limit of $\{u_n\}$.

We now prove that the seqn $\{u_n\}$ converges to l.

Let us choose $\epsilon > 0$. There exists a natural number K such that

$|u_m - u_n| < \epsilon/2 \quad \forall\ m, n \geq K$ ----(1)

Since l is a subsequential limit of $\{u_n\}$, there exists a natural number $q > K$ such that $|u_q - l| < \epsilon/2$.

Since $q > K$, from (1) $\quad |u_q - u_n| < \epsilon/2 \quad \forall\ n \geq K$.

Now $|u_n - l| = |u_n - u_q| + |u_q - l| < \epsilon/2 + \epsilon/2 \quad \forall\ n \geq K$.

i.e $\quad |u_n - l| < \epsilon \quad \forall\ n \geq K$.

$\Rightarrow \quad \lim u_n = l. \Rightarrow \{u_n\}$ is convergent.

* A Convergent seqn is a cauchy seqn.

Proof:- Let $\{u_n\}$ be a convergent seqn and let $\lim u_n = l$.

For a pre-assigned $\epsilon > 0$, $\exists$ a $K \in \mathbb{N}$ s.t

$$|u_n - l| < \epsilon/2 \quad \forall\, n \geq K.$$

If m, n be natural numbers $\geq K$, then

$$|u_m - l| < \epsilon/2 \text{ and } |u_n - l| < \epsilon/2.$$

Now $|u_m - u_n| \leq |u_m - l| + |l - u_n| < \epsilon/2 + \epsilon/2 \quad \forall\, m, n \geq K.$

i.e. $|u_m - u_n| < \epsilon \quad \forall\, m, n \geq K.$

$\Rightarrow \{u_n\}$ is a cauchy seqn.

Example:- Prove that the seqn $\{\frac{1}{n}\}$ is a cauchy sequence.

Let $u_n = \frac{1}{n}$. Let us choose a positive ϵ. There $\exists$ $K \in \mathbb{N}$, such that $\frac{2}{K} < \epsilon$.

$$|u_m - u_n| = \left|\frac{1}{m} - \frac{1}{n}\right| < \frac{1}{m} + \frac{1}{n} \leq \frac{1}{K} + \frac{1}{K} \quad \text{if } m, n \geq K$$
$$< \epsilon \quad \text{if } m, n \geq K.$$

$\Rightarrow \{u_n\}$ is a cauchy seqn.

* Prove that seqn $\{(-1)^n\}$ is not a cauchy sequence.

Let $u_n = (-1)^n$. Then

$$|u_m - u_n| = |(-1)^m - (-1)^n|$$

Now, $|u_m - u_n| = 0$ if m, n are both odd or both even,
$= 2$ if one of m, n is odd and the other is even.

Let us choose $\epsilon = \frac{1}{2}$. Then it is not possible to find a natural number K such that $|u_m - u_n| < \epsilon \ \forall\, m, n \geq K$.

Hence $\{u_n\}$ is not a cauchy seqn.

* Cauchy's theorems on limits.

Theorem :- If $\lim u_n = l$, then $\lim \frac{u_1+u_2+\cdots+u_n}{n} = l$.

proof:- Case 1:- $l=0$

$\because \{u_n\}$ is a convergent seqn, it is bounded. Therefore, there exists a positive number B such that $|u_n| < B \ \forall n \in \mathbb{N}$.

Let $\epsilon > 0$. Since $\lim u_n = 0$, $\exists$ a $k_1 \in \mathbb{N}$ s.t $|u_n| < \epsilon/2 \ \forall n \geq k_1$.

Now
$$\left|\frac{u_1+u_2+\cdots+u_n}{n}\right| \leq \left|\frac{u_1+u_2+\cdots+u_{k_1-1}}{n}\right| + \left|\frac{u_{k_1}+u_{k_1+1}+\cdots+u_n}{n}\right|$$
$$\leq \left|\frac{|u_1|+|u_2|+\cdots+|u_{k_1-1}|}{n}\right| + \left|\frac{|u_{k_1}|+|u_{k_1+1}|+\cdots+|u_n|}{n}\right|$$
$$< \frac{B(k_1-1)}{n} + \frac{n-(k_1-1)}{n}\cdot\epsilon/2 \quad \forall n \geq k_1$$
$$< \frac{Bk_1}{n} + \epsilon/2 \quad \forall n \geq k_1$$

$\because \lim \frac{1}{n} = 0$, $\exists$ $k_2 \in \mathbb{N}$ s.t $\frac{Bk_1}{n} < \epsilon/2 \ \forall n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then $\left|\frac{u_1+u_2+\cdots+u_n}{n}\right| < \epsilon \ \forall n \geq k$.

$$\Rightarrow \lim \frac{u_1+u_2+\cdots+u_n}{n} = 0.$$

Case 2:- $l \neq 0$.

Let $v_n = u_n - l$. Then $\lim v_n = 0$

Now,
$$\frac{u_1+u_2+\cdots+u_n}{n} - l = \frac{v_1+v_2+\cdots+v_n}{n}.$$

By case 1; $\lim \frac{v_1+v_2+\cdots+v_n}{n} = 0$. Therefore,
$$\lim \frac{u_1+u_2+\cdots+u_n}{n} = l.$$

Note:- The converse of this thm is not true.

Let $u_n = (-1)^n$.

$\lim \frac{u_1+u_2+\cdots+u_n}{n} = 0$ but the seqn $\{u_n\}$ is not Convergent

* Corollary:- If $\lim u_n = l$, where $u_n > 0 \ \forall n$ and $l \neq 0$, then
$$\lim \sqrt[n]{u_1 u_2 \cdots u_n} = l.$$

Since each u_n is positive and $\lim u_n = l > 0$, the seqn $\{\log u_n\}$ converges to $\log l$. ~~by the~~ [we know that if $\{u_n\}$ be s.t $u_n > 0 \ \forall n \in \mathbb{N}$. and $\lim u_n = c > 0$ $\lim \log u_n = \log c$]

or, ~~log log~~

Therefore, $\lim \frac{\log u_1 + \log u_2 + \cdots + \log u_n}{n} = \log l$

or $\lim \log \sqrt[n]{(u_1 u_2 \cdots u_n)} = \log l.$

$\Rightarrow \lim \sqrt[n]{(u_1 u_2 \cdots u_n)} = l$

Example:- 1. Prove that $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$

Let $u_n = \frac{1}{n}$. Then $\lim u_n = 0$.

By Cauchy's theorem, $\lim \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$

2. Prove that $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1.$

Let $u_n = \sqrt[n]{n}$. Then $\lim u_n = 1.$

By Cauchy's theorem, $\lim \frac{1+\sqrt{2}+\sqrt[3]{3}+\cdots+\sqrt[n]{n}}{n} = 1.$

Thm:- If $u_n > 0 \ \forall n \in \mathbb{N}$ and $\lim \frac{u_{n+1}}{u_n} = l \ (\neq 0)$ then $\lim \sqrt[n]{u_n} = l.$

Proof:- Let $v_1 = u_1$, $v_2 = \frac{u_2}{u_1}$, $v_3 = \frac{u_3}{u_2}, \ldots, v_n = \frac{u_n}{u_{n-1}}, \ldots$

Then $v_n > 0 \ \forall n \in \mathbb{N}$ and $\lim v_n = l > 0$.

$\Rightarrow \log v_n = \log l.$

By 1st thm $\lim \frac{\log v_1 + \log v_2 + \cdots + \log v_n}{n} = \log l.$

$\Rightarrow \lim \log \sqrt[n]{(v_1 v_2 \cdots v_n)} = \log l.$

$\Downarrow$

$\log (v_1 v_2 \cdots v_n)^{\frac{1}{n}}$

$\Rightarrow \lim \sqrt[n]{u_n} = l.$

* **Example:-** Prove that $\sqrt[n]{n} = 1.$

Let $u_n = n$. Then $u_n > 0 \ \forall n \in \mathbb{N}$ $\lim \frac{u_{n+1}}{u_n} = 1 > 0.$

$\Rightarrow \sqrt[n]{n} = 1.$

* Prove that $\lim \frac{\{(n+1)(n+2)\cdots(2n)\}^{1/n}}{n} = 4/e$

$u_n = \frac{(n+1)(n+2)\cdots 2n}{n^n}$. Then $u_n > 0$ $\forall n \in \mathbb{N}$ and

$$\lim \frac{u_{n+1}}{u_n} = \frac{2(2n+1)}{n+1} \cdot \frac{1}{(1+\frac{1}{n})^n} = 4/e > 0.$$

$$\Rightarrow \sqrt[n]{u_n} = 4/e .$$

* <u>Assignment</u> :-

1. Find $\lim \frac{(n!)^{1/n}}{n}$.
2. Find $\frac{1}{n}\{(2n+1)(2n+2)\cdots(2n+n)\}^{\frac{1}{n}}$.
3. a) Show that if $\{a_n\}$ is cauchy then $\{a_n^2\}$ is also cauchy.

   b) Give an example of a cauchy seqn $\{a_n^2\}$ such that $\{a_n\}$ is not cauchy.
4. Prove that the seqn $x_n = \frac{2n+1}{3n}$ is a cauchy seqn.
5. Sum of two cauchy sequences is again a cauchy seqn.
6. If $\{u_n\}$ be a cauchy seqn in $\mathbb{R}$ having a subsequence converging to a real number l, then prove that $\lim u_n = l$.