Rule's For finding P.I

Linear Differential eqns with Constant Coefficient can be written as $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = X$, where X is a function of x, $a_1, a_2, \ldots a_n$ are constant.

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)\, y = X, \quad D = \frac{d}{dx}$$

$$F(D)\, y = X.$$

$$y = \frac{1}{F(D)} \cdot X.$$

$$\therefore y_{P.I} = \frac{1}{F(D)} \cdot X.$$

Case :- When X is $\cos ax / \sin ax$

(a) Express $F(D)$ as function of D^2, say $\phi(D^2)$, then replace D^2 by $-a^2$, if $\phi(-a^2) \neq 0$.

$$\therefore P.I = \frac{1}{F(D)} \cos ax / \sin ax = \frac{1}{\phi(D^2)} \cos ax / \sin ax = \frac{1}{\phi(-a^2)} \cos ax / \sin ax$$

Example :- $P.I = \frac{1}{D^2 - 2D + 1} \cos 3x$

$$= \frac{1}{-3^2 - 2D + 1} \cos 3x = -\frac{1}{2} \frac{1}{D+4} \cos 3x$$

$$= -\frac{1}{2} \frac{D-4}{D^2 - 4^2} \cos 3x$$

$$= -\frac{1}{2} (D-4) \frac{1}{-3^2 - 4^2} \cos 3x$$

$$= \frac{1}{50} (D-4) \cos 3x$$

$$= \frac{1}{50} \left[-3\sin 3x - 4\cos 3x \right].$$

(b) If $\phi(-a^2) = 0$, then we can proceed as follows:

$$P.I = \frac{1}{D^2 + a^2} \sin ax$$

$$= \text{Imaginary part of } \frac{1}{D^2 + a^2} e^{iax} \quad \ldots (1)$$

Now $\frac{1}{D^2 + a^2} e^{iax}$

$$= e^{iax} \cdot \frac{1}{(D+ia)^2 + a^2} \cdot e^{0x}$$

$$= e^{iax} \cdot \frac{1}{D^2 + 2aiD} \cdot e^{0.x}$$

$$= e^{iax} \cdot \frac{1}{D(D + 2ai)} \cdot e^{0.x}$$

$$\frac{1}{f(D)} e^{ax}$$

$$= e^{ax} \cdot \frac{1}{F(D+a)} \cdot e^{0.x}$$

$$e^{ix} = \cos x + i \sin x$$

$\therefore \sin x = $ imaginary part of e^{ix}

$$= e^{iax} \cdot \frac{1}{D} \frac{1}{0+2ia} \cdot e^{0.x}$$

$$= \frac{e^{iax}}{2ai} \cdot \frac{1}{D} \cdot 1 = \frac{e^{aix}}{2ai} \cdot x = \frac{x}{2ai}(\cos ax + i\sin ax)$$

$$= \frac{x}{2a}\sin ax + \frac{x\cos ax}{2ai}$$

$$= \frac{x}{2a}\sin ax - i\frac{x}{2a}\cos ax$$

$\therefore$ From (1) P.I $=$ imaginary part of $\left(\frac{x}{2a}\sin ax - i\frac{x}{2a}\cos ax\right)$

$$= -\frac{x}{2a}\cos ax.$$

Similarly, $\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax.$

* __Case__ :- When X is x^m or a polynomial of degree m, m being (+ve) integer.

__Example__ :- $(D^3 - D^2 - 6D)y = x^2 + 1$

The auxiliary equation is $m^3 - m^2 - 6m = 0$

$$\Rightarrow m(m^2 - m - 6) = 0$$

$$\Rightarrow m(m-3)(m+2) = 0$$

$$m = 0, 3, -2$$

$y_{C.F} = C_1 e^{0.x} + C_2 e^{3x} + C_3 e^{-2x}$

$$P.I = \frac{1}{D^3 - D^2 - 6D}(x^2+1) = \frac{1}{-6D\left(1 + \frac{D}{6} - \frac{D^2}{6}\right)}(x^2+1)$$

$$+\frac{D^2}{6} + \frac{D^2}{36}$$

$$= -\frac{1}{6D}\left[\left(1 + \frac{D}{6} - \frac{D^2}{6}\right)^{-1}\right](x^2+1)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 - \right](x^2+1)$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{7}{36}D^2 + \cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[x^2 + 1 - \frac{1}{6}(2x) + \frac{7}{36} \times 2 \cdots\right]$$

$$= -\frac{1}{6D}\left[x^2 + 1 - \frac{x}{3} + \frac{7}{16}\right]$$

$$= -\frac{1}{6D}\left[x^2 - \frac{1}{3}x + \frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right]$$

* Case:- When X is $e^{ax}.V$, V is any function of x.

$$P.I = \frac{1}{f(D)} \cdot e^{ax}.V = e^{ax}. \frac{1}{f(D+a)} . V$$

Example:- $(D^2+3D+2)y = e^{2x}\sin x$

Auxiliary eqn $m^2+3m+2=0$

$m = -2, -1$

$y_{C.F} = c_1 e^{-2x} + c_2 e^{-x}$

$$P.I = \frac{1}{D^2+3D+2} e^{2x}\sin x = e^{2x}. \frac{1}{(D+2)^2+3(D+2)+2} . \sin x$$

$$= e^{2x} \frac{1}{D^2+7D+12} \sin x$$

$$= e^{2x}. \frac{1}{-1^2+7D+12} \sin x$$

$$= e^{2x} \frac{(11-7D)}{(11-7D)(11+7D)} \sin x$$

$$= e^{2x}. (11-7D) \frac{1}{121-49D^2} \sin x$$

$$= e^{2x} (11-7D). \frac{1}{121-49(-1^2)} \sin x$$

$$= e^{2x} (11-7D). \frac{1}{121+49} \sin x$$

$$= \frac{e^{2x}}{170} (11\sin x - 7\cos x)$$

* Case:- When X is $x.V$, V is any function of x.

Then $$P.I = \frac{1}{F(D)}.(x.V) = x\frac{1}{F(D)} V - \frac{F'(D)}{[F(D)]^2} V.$$

Example:- Solve $(D^2-2D+1)y = x\sin x$

Auxiliary eqn $m^2-2m+1=0$, $m=1,1$

$y_{C.F} = (c_1+c_2x)e^{x}$.

$$P.I = \frac{1}{D^2-2D+1} x\sin x = x\frac{1}{D^2-2D+1}\sin x - \frac{2D-2}{(D^2-2D+1)^2}\sin x$$

$$= x\frac{1}{-1-2D+1}\sin x - \frac{2D-2}{(-1-2D+1)^2}\sin x$$

$$= -\frac{1}{2}x.\frac{1}{D}\sin x - \frac{1}{2}\frac{1}{D^2}(D-1)\sin x$$

$$= \frac{1}{2}x\cos x - \frac{1}{2}\frac{1}{D^2}(\cos x - \sin x)$$

$$= \frac{1}{2}x\cos x - \frac{1}{2}(-\cos x - \sin x)$$

Assignment - 1

* Find the solution of $\frac{d^2y}{dx^2} + 4y = 8\cos 2x$, given that $y = 0$ and $\frac{dy}{dx} = 0$, when $x = 0$.

* Solve $(D^2 + 4)y = \sin 2x$, given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 2$.

* $(D^2 - 4D + 4)y = x^2$

* $(D^3 + 3D^2 + 2D)y = x^2$

* $(D^2 + 2D + 1)y = 2x + x^2$

* Solve $(D^2 - 2D + 1)y = x^2 e^{3x}$

* Solve $(D^2 - 4D - 5)y = xe^{-x}$, given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.

* Solve $(D^2 + 2D + 1)y = x\cos x$

* Solve $(D^2 - 1)y = xe^x + \cos^2 x$.

* Solve $(D^2 + 1)y = x^3 + e^x \sin x$